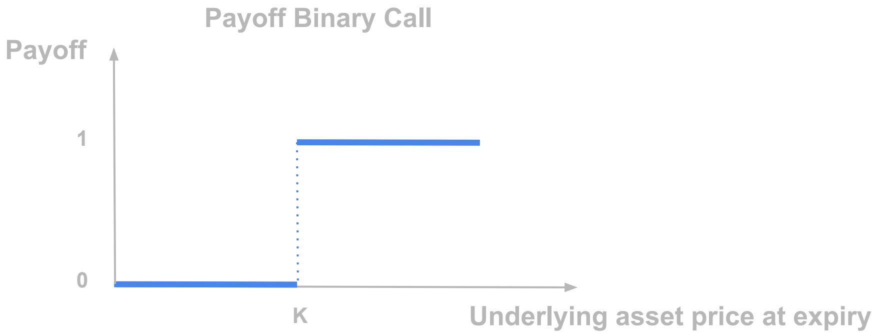
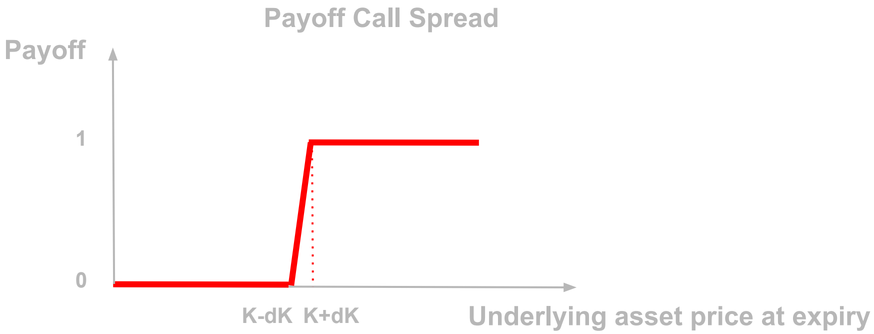
Binary Options: Replication and Skew Sensitivity

Binary Payoff and Replication

A binary (or digital) cash-or-nothing call option has the following payoff, with K the strike price and S the underlying asset:



This payoff can be replicated by a call spread with strikes K1 = K - dK and K2 = K + dK, and a quantity = 1 / (2 x dK) with dK small enough



Payoff binary call option

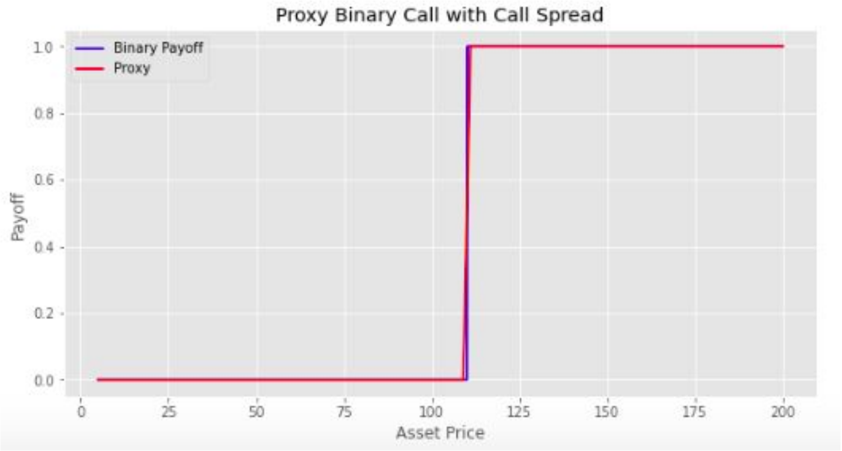
$$\begin{cases} 0 & \text{if} \quad S < K \\ 1 & \text{if} \quad S \geq K \end{cases}$$

$$\text{Binary Call} = \lim_{dK \to 0} \frac{1}{2 \cdot dK} \cdot \left(Call(K - dK) - Call(K + dK)\right)$$

Binary Payoff and Replication

```python
def PayoffCallSpread(S, K1, K2):
    call1 = np.maximum(S - K1, 0)
    call2 = np.maximum(S - K2, 0)
    return call1 - call2
```

```python
K = 110

S = np.arange(5,200,.1)
dK = 1

PayOffBinary = [0 if x < K else 1 for x in S]
PayOffProxy = [1 / (2 * dK) * PayoffCallSpread(x , K - dK, K + dK) for x in S]

plt.rcParams["figure.figsize"] = (10,5)
plt.plot(S,PayOffBinary, color = 'b')
plt.plot(S, PayOffProxy, color = 'r')
plt.ylabel('Payoff')
plt.xlabel('Asset Price')
plt.legend(['Binary Payoff', 'Proxy'])
plt.title('Proxy Binary Call with Call Spread')
plt.show;
```



Binary Call Price in Black-Scholes Model

If we assume that the only changing variable is the strike price, the price of a binary call is equal to the opposite of the derivatives of the call price with respect to the strike price:

$$\text{Binary Call} = \lim_{dK \to 0} \frac{1}{2 \cdot dK} \cdot (Call(K - dK) - Call(K + dK))$$

$$C_{\text{binary}} = -\frac{\partial Call}{\partial K}$$

It can be proxy by finite difference method:

$$C_{\text{binary}} \approx \frac{Call(S,K-\delta K,\sigma,r,T) - Call(S,K+\delta K,\sigma,r,T)}{2 \cdot \delta K}$$

In the Black-Scholes framework there is a closed form solution for the price of a call option:

$$Call = S \cdot N(d_1) - K \cdot e^{-r \cdot T} \cdot N(d_2)$$

$$\text{with } d_2 = \frac{\ln\left(\frac{S_t}{K}\right) - \left(r - \frac{\sigma^2}{2}\right) \cdot (T-t)}{\sigma \cdot \sqrt{T-t}} \qquad d_1 = d_2 + \sigma \cdot \sqrt{T}$$

$$N(x) = \frac{1}{\sqrt{2 \cdot \pi}} \cdot \int_{-\infty}^{x} e^{-\frac{u^2}{2}} du$$

The price of the binary call is equal to the probability that the option will be exercised times the discount factor:

$$C_{\text{binary}} = e^{-r \cdot T} \cdot N(d_2)$$

Binary Call Price in Black-Scholes Model

```
def CallPrice(S, K, r, sigma, T):
    d1 = (math.log(S / K) + (r + .5 * sigma**2) * T) / (sigma * T**.5)
    d2 = d1 - sigma * T**.5
    n1 = norm.cdf(d1)
    n2 = norm.cdf(d2)
    DF = math.exp(-r * T)
    price=S * n1 - K * DF * n2
    return price

def PutPrice(S, K, r, sigma, T):
    d1 = (math.log(S / K) + (r + .5 * sigma**2) * T) / (sigma * T**.5)
    d2 = d1 - sigma * T**.5
    n1 = norm.cdf(-d1)
    n2 = norm.cdf(-d2)
    DF = math.exp(-r * T)
    price= K * DF * n2 - S * n1
    return price

def DigitalCallPriceBS(S, K, r, sigma, T):
    d1 = (math.log(S / K) + (r + .5 * sigma**2) * T) / (sigma * T**.5)
    d2 = d1 - sigma * T**.5
    n2 = norm.cdf(d2)
    DF = math.exp(-r * T)
    price= DF * n2
    return price

def DigitalPutPriceBS(S, K, r, sigma, T):
    d1 = (math.log(S / K) + (r + .5 * sigma**2) * T) / (sigma * T**.5)
    d2 = d1 - sigma * T**.5
    n2 = norm.cdf(-d2)
    DF = math.exp(-r * T)
    price= DF * n2
    return price
```

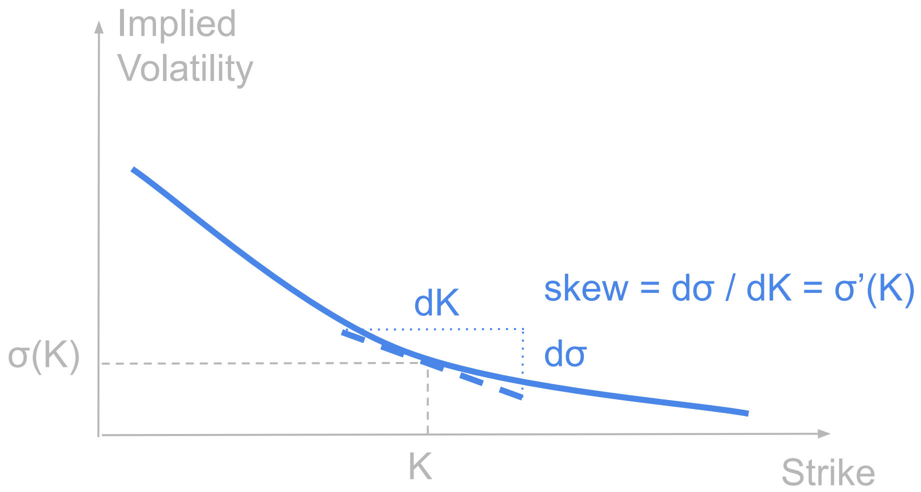
```
#parameters
S0 = 100 #asset price
K = 110 #strike price
r= .05 #risk-free interest rate
sigma = .3 #volatility
T = .25 #time to maturity
dK = 0.1

pricedigital = DigitalCallPriceBS(S0, K, r, sigma, T)
priceproxy = 1/(2 * dK) * (CallPrice(S0, K - dK, r, sigma, T) - \
                           CallPrice(S0, K + dK, r, sigma, T))
print("Binary Call Price:" + str(np.round(pricedigital, 3)))
print("Proxy Call Spread:" + str(np.round(priceproxy, 3)))
```

```
Binary Call Price:0.262
Proxy Call Spread:0.262
```

What if the implied volatility is not constant?

Skew Sensitivity of a Binary Call Option

We assume that the implied volatility is no more constant, it is a function of the strike price. σ(K) is the implied volatility of a call option at the strike K.



$$Call(K) = Call_{BS}(K, \sigma(K))$$

$$C_{\text{binary}} = -\frac{\partial Call}{\partial K}$$

$$C_{\text{binary}} = \underbrace{-\frac{\partial Call_{BS}}{\partial K}}_{\text{Black-Scholes price}} \underbrace{-\frac{\partial Call_{BS}}{\partial \sigma}} \cdot \sigma'(K)$$

Black-Scholes price

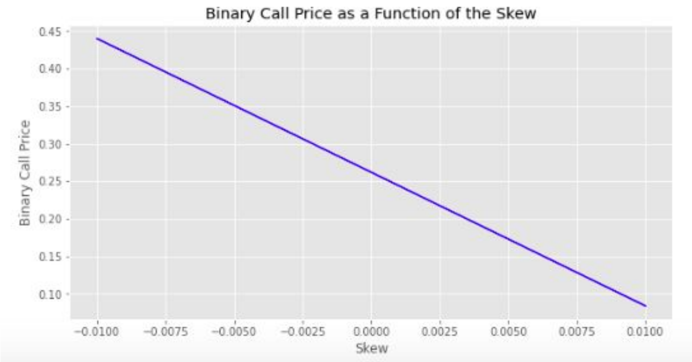
$$e^{-r \cdot T} \cdot N(d_2)$$

$$\nu = S \cdot e^{-\frac{d_1^2}{2}} \cdot \sqrt{\frac{T}{2 \cdot \pi}}$$

> 0 when $\sigma'(K) < 0$

Skew Sensitivity of a Binary Call Option

```
#parameters
S0 = 100 #asset price
K = 110 #strike price
r= .05 #risk-free interest rate
sigma = .3 #volatility
T = .25 #time to maturity
skew = np.arange(-.01, .011, .001)

PriceBS = DigitalCallPriceBS(S0, K, r, sigma, T)
d1 = (math.log(S0 / K) + (r + .5 * sigma**2) * T) / (sigma * T**.5)
vega = S0 * math.exp(- .5 * d1**2) * (T / (2 * np.pi))**.5

priceproxy = [PriceBS - vega * s for s in skew]
plt.rcParams["figure.figsize"] = (10,5)
plt.plot(skew, priceproxy, color = 'b')
plt.ylabel('Binary Call Price')
plt.xlabel('Skew')
plt.title('Binary Call Price as a Function of the Skew')
plt.show;
```



$$Call(K) = Call_{BS}(K, \sigma(K))$$

$$C_{\text{binary}} = -\frac{\partial Call}{\partial K}$$

$$C_{\text{binary}} = \underbrace{-\frac{\partial Call_{BS}}{\partial K}}_{\text{Black-Scholes price}} \underbrace{-\frac{\partial Call_{BS}}{\partial \sigma}}_{} \cdot \sigma'(K)$$

Black-Scholes price

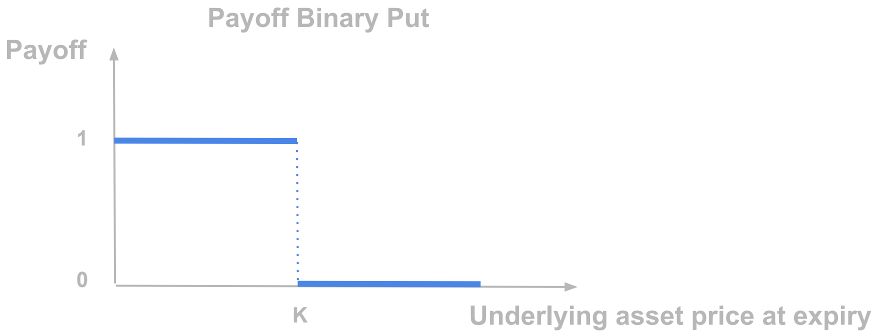
$$e^{-r \cdot T} \cdot N(d_2)$$

$$\nu = S \cdot e^{-\frac{d_1^2}{2}} \cdot \sqrt{\frac{T}{2 \cdot \pi}}$$

> 0 when σ’(K) < 0

Skew Sensitivity of a Binary Put Option

A binary (or digital) cash-or-nothing put option has the following payoff, with K the strike price and S the underlying asset:



And it can be replicated by a put spread:

$$\text{Binary Put} = \lim_{dK \to 0} \frac{1}{2 \cdot dK} \cdot (Put(K + dK) - Put(K - dK))$$

$$P_{\text{binary}} = \frac{\partial P}{\partial K}$$

$$Put(K) = Put_{BS}(K, \sigma(K))$$

$$P_{\text{binary}} = \frac{\partial P}{\partial K}$$

$$P_{\text{binary}} = \underbrace{\frac{\partial Put_{BS}}{\partial K}}_{\text{Black-Scholes price}} + \underbrace{\frac{\partial Put_{BS}}{\partial \sigma}}_{\nu} \cdot \sigma'(K)$$

Black-Scholes price

$$e^{-r \cdot T} \cdot N(-d_2)$$

$$\nu = S \cdot e^{-\frac{d_1^2}{2}} \cdot \sqrt{\frac{T}{2 \cdot \pi}}$$

< 0 when σ'(K) < 0

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