The Vasicek Model

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The Vasicek model is a mathematical model used in finance to describe the movement of interest rates over time.
It was developed by Oldrich Vasicek in 1977[1].
It describes the evolution of interest rates by assuming that the short-term interest rate follows a mean-reverting stochastic process.

[1]Reference: Vasicek, O. (1977), "An equilibrium characterization of the term structure", Journal of Financial Economics.

The Stochastic Differential Equation

The instantaneous interest rate r_t follows the following stochastic differential equation (SDE):

$$dr_t = a \cdot (b - r_t) \cdot dt + \sigma \cdot dW_t$$

Speed of reversion $a>0$

Long-term mean

Instantaneous volatility

Simulation of Interest Rates with the Vasicek Model

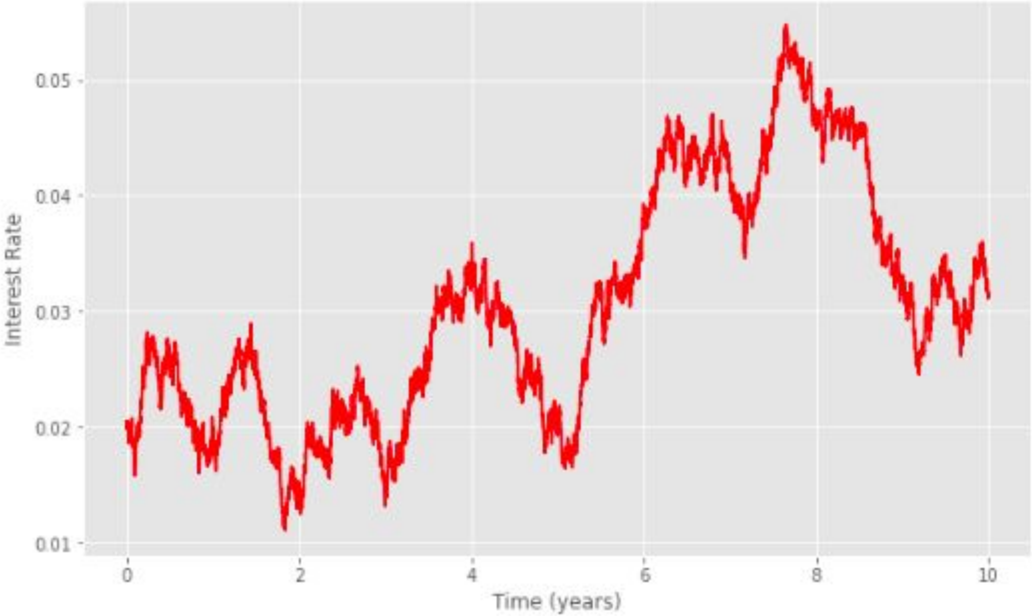
$r_0 = 2\%$

$a = 0.5$

$b = 3\%$

$\sigma = 1\%$

$T = 10y$



Simulation of Interest Rates with the Vasicek Model

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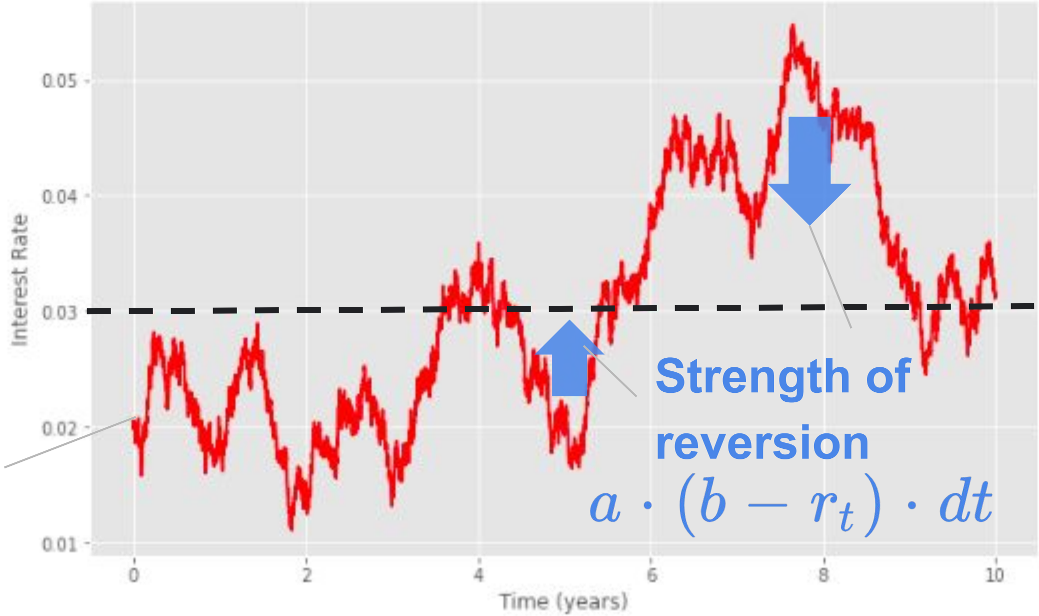
Initial value r_0



Long-term mean (b)

Strength of reversion

$$a \cdot (b - r_t) \cdot dt$$

Solution of the SDE

$$r_t = r_0 \cdot e^{-a \cdot t} + b \cdot (1 - e^{-a \cdot t}) + \sigma \cdot e^{-a \cdot t} \cdot \int_0^t e^{a \cdot s} \cdot dW_s$$

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Demo:

Itô Lemma with $x_t = r_t \cdot e^{a \cdot t}$

$$dx_t = dr_t \cdot e^{a \cdot t} + r_t \cdot a \cdot e^{a \cdot t} \cdot dt$$

$$dr_t = a \cdot (b - r_t) \cdot dt + \sigma \cdot dW_t$$

$$dx_t = a \cdot b \cdot e^{a \cdot t} \cdot dt + \sigma \cdot e^{a \cdot t} \cdot dW_t$$

$$x_t - x_0 = a \cdot b \cdot \int_0^t e^{a \cdot s} \cdot ds + \sigma \cdot \int_0^t e^{a \cdot s} \cdot dW_s$$

$$r_t = r_0 \cdot e^{-a \cdot t} + b \cdot (1 - e^{-a \cdot t}) + \sigma \cdot e^{-a \cdot t} \cdot \int_0^t e^{a \cdot s} \cdot dW_s$$

Solution of the SDE

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r_t follows a normal distribution.

$$r_t \sim N\left(r_0 \cdot e^{-a \cdot t} + b \cdot (1 - e^{-a \cdot t}), \frac{\sigma^2}{2 \cdot a} \cdot (1 - e^{-2 \cdot a \cdot t})\right)$$

Solution of the SDE

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r_t follows a normal distribution. It admits a stationary probability distribution.

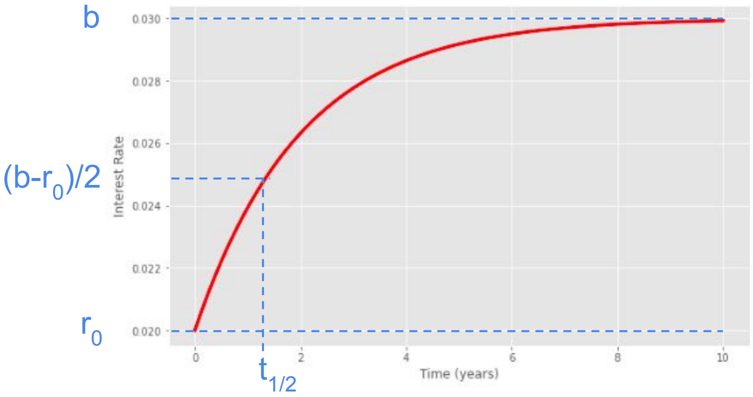
$$r_t \sim N\left(\underbrace{r_0 \cdot e^{-a \cdot t} + b \cdot \left(1 - e^{-a \cdot t}\right)}_{\underset{t \to +\infty}{\longrightarrow} b}, \underbrace{\frac{\sigma^2}{2 \cdot a} \cdot \left(1 - e^{-2 \cdot a \cdot t}\right)}_{\underset{t \to +\infty}{\longrightarrow} \frac{\sigma^2}{2 \cdot a}}\right)$$

Half-Life of the Mean-Reversion

It is the average time it take to be half-way back to the mean.

$$r_0 \cdot e^{-a \cdot t_{\frac{1}{2}}} + b \cdot \left(1 - e^{-a \cdot t_{\frac{1}{2}}}\right) - r_0 = \frac{b - r_0}{2}$$

$$\Rightarrow \quad \boxed{t_{\frac{1}{2}} = \frac{\ln(2)}{a}}$$



r_0 = 2%, a = 0.5 , b = 3%

The higher the speed of reversion the smaller the half-life.

Simulated Interest Rates Paths

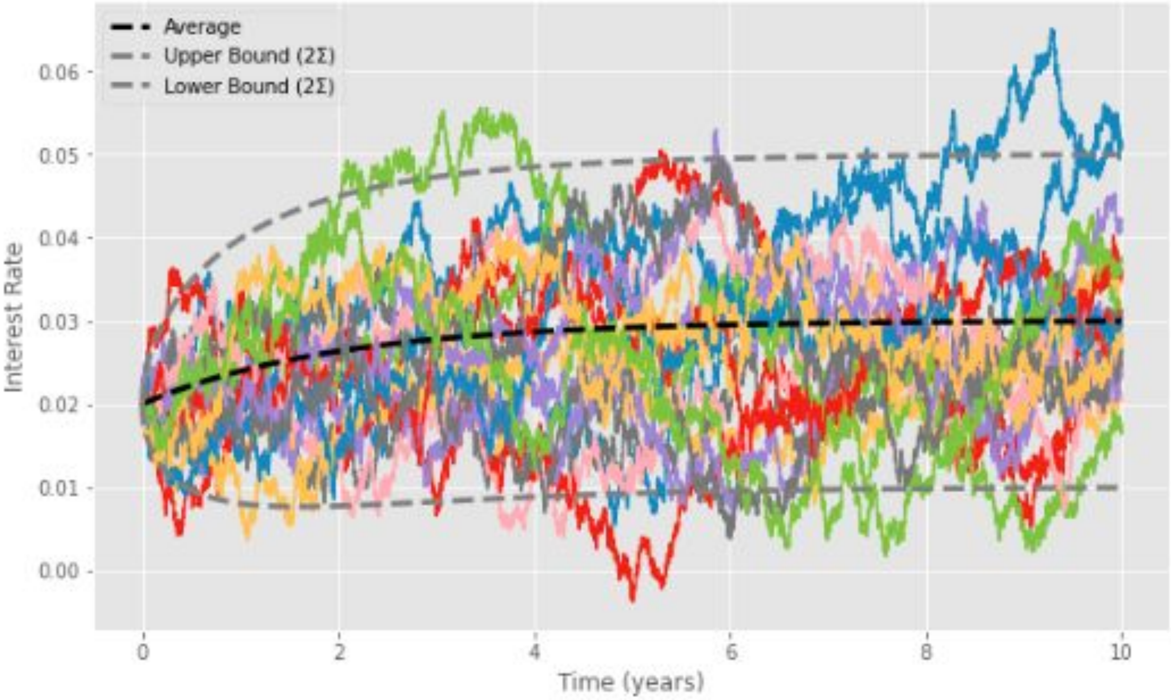
$r_0 = 2\%$

$a = 0.5$

$b = 3\%$

$\sigma = 1\%$

$T = 10y$



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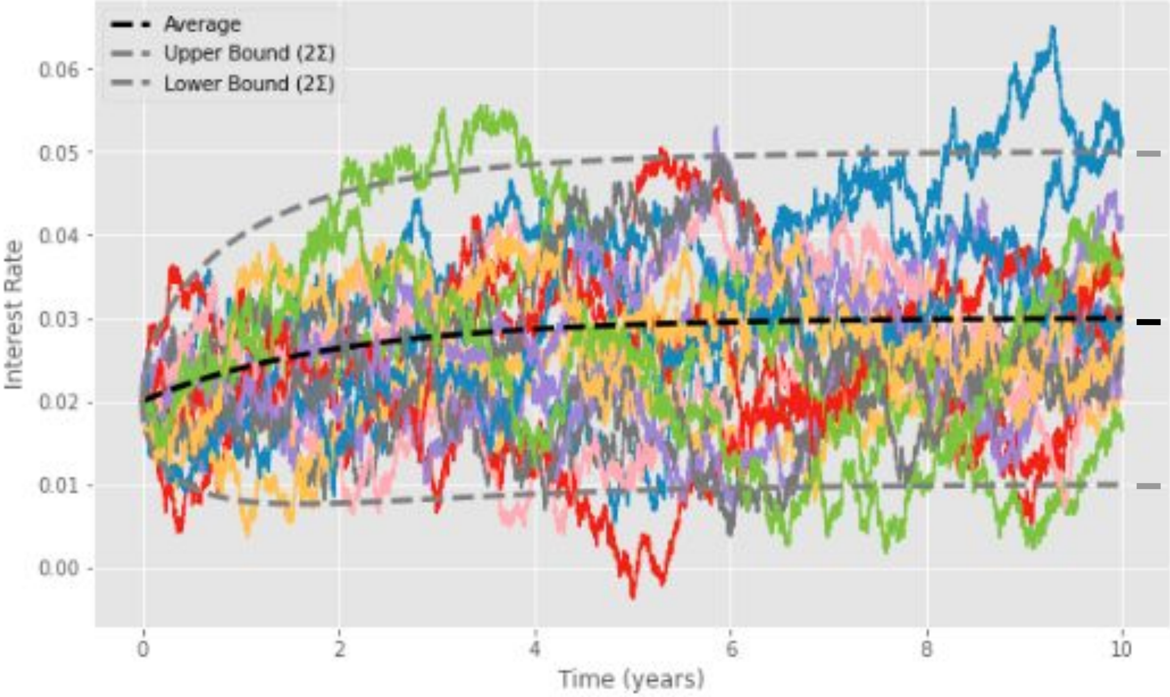

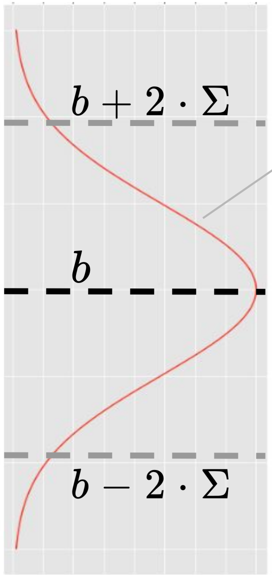

$N(b, \Sigma^2)$

$$\Sigma = \frac{\sigma}{\sqrt{2 \cdot a}}$$

Zero-Coupon Bond Pricing

The price of a zero-coupon bond at the date t with maturity T has the following expression in the Vasicek model under the no-arbitrage assumption:

$$P(t,T) = e^{A(t,T) - B(t,T)\cdot r(t)}$$ with $$A(t,T) = \left(b - \frac{\sigma^2}{2\cdot a^2}\right)\cdot\left(B(t,T) - (T-t)\right) - \frac{\sigma^2}{4\cdot a}\cdot B(t,T)^2$$

$$B(t,T) = \frac{1 - e^{-a\cdot(T-t)}}{a}$$

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Demo:

$$P(t,T) = E^Q\left(e^{-\int_t^T r_u du} \middle| F_t\right)$$

$\int_t^T r_u du$ ~$N(m(t,T), v(t,T))$

Conditionally to F_t. m(t,T) and v(t,T) are the average and the variance of the integral of r_t and can be directly calculated from the expression of r_t.

$$P(t,T) = e^{-m(t,T) + \frac{1}{2}\cdot v(t,T)}$$

$$m(t,T) = b\cdot(T-t) + (r_t - b)\cdot\frac{1-e^{-a\cdot(T-t)}}{a}$$

$$v(t,T) = \frac{\sigma^2}{a^2}\cdot\left((T-t) - 2\cdot\frac{1-e^{-a\cdot(T-t)}}{a} + \frac{1-e^{-2\cdot a\cdot(T-t)}}{2\cdot a}\right)$$

Limits

Gaussian assumption which is not realistic.

Possibility to have negative values, which can be problematic, typically if we want to model default intensity and credit spreads or in pre-crisis non-negative interest-rates world.

The flexibility of the model is limited, it is typically unable to reproduce all zero-coupon curve shapes observed in the market.

It is a single factor model, with a constant volatility parameter and it does not incorporate jumps.

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