The Cox-Ingersoll-Ross Model

The Cox-Ingersoll Ross Model

The Cox-Ingersoll-Ross (CIR) model is a stochastic interest rate model used in finance to describe the evolution of interest rates.

The model was introduced in 1985[1] as an alternative to the Vasicek model. It assumes that the short-term interest rate follows a mean-reverting stochastic process, it does not allow negative interest rates while preserving analytical solution for bond pricings.

It is also used in the popular stochastic volatility Heston model[2] to model the stochastic variance.

References:

[1]Cox, J.C., Ingersoll, J.E. and Ross, S.A. (1985), "A Theory of the Term Structure of Interest Rates", Econometrica.

[2]See Heston, Steven L. (1993), “A closed-form solution for options with stochastic volatility with applications to bond and currency options”.

The Stochastic Differential Equation

The instantaneous interest rate r_t follows the following stochastic differential equation (SDE):

$$dr_t = \boxed{a \cdot (b - r_t)} \cdot dt + \sigma \cdot \boxed{\sqrt{r_t}} \cdot dW_t$$

Speed of reversion
a > 0

Long-term mean
b > 0

Volatility

The Stochastic Differential Equation

The instantaneous interest rate r_t follows the following stochastic differential equation (SDE):

Similar to Vasicek model

Avoids to have negative interest rates

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Speed of reversion
a > 0

Long-term mean
b > 0

Volatility

The Feller Condition

The instantaneous interest rate r_t follows the following stochastic differential equation (SDE):

$$dr_t = a \cdot (b - r_t) \cdot dt + \sigma \cdot \sqrt{r_t} \cdot dW_t$$

If the following condition is satisfied, then r_t is strictly positive.

This is known as the Feller condition.

$$2 \cdot a \cdot b > \sigma^2$$

Simulation of Interest Rates with the CIR Model

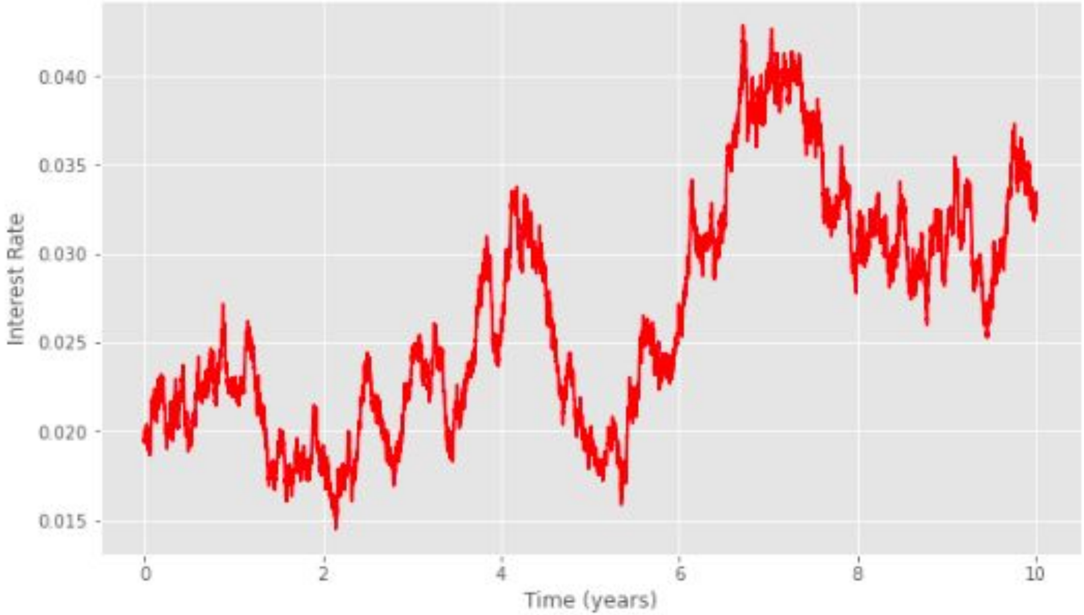
$r_0 = 2\%$

$a = 0.5$

$b = 3\%$

$\sigma = 5\%$

$T = 10y$



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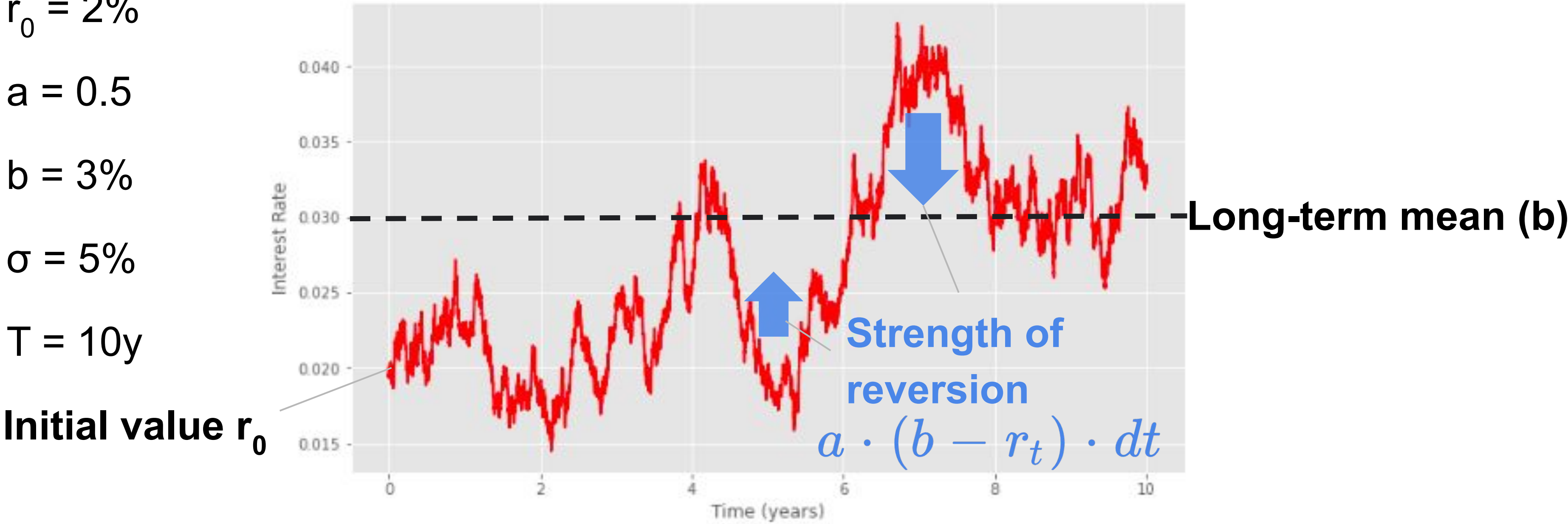

Probability Density Function

The probability density function of the future value r_s at the future time s conditionally to its current value r_t at the current time t is given by:

$$f(r_s, s; r_t, t) = c \cdot e^{-(u+v)} \cdot \left(\frac{v}{u}\right)^{\frac{q}{2}} \cdot I_q\left(2 \cdot (u \cdot v)^{\frac{1}{2}}\right)$$

Where:

$$c = \frac{2 \cdot a}{\sigma^2 \cdot \left(1 - e^{-a \cdot (s-t)}\right)}$$

$$u = c \cdot r_t \cdot e^{-a \cdot (s-t)}$$

$$v = c \cdot r_s$$

$$q = \frac{2 \cdot a \cdot b}{\sigma^2} - 1$$

I_q is the modified Bessel function of the first kind of order q.

Reference: Cox, J.C., Ingersoll, J.E. and Ross, S.A. (1985), "A Theory of the Term Structure of Interest Rates", Econometrica.

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We can calculate the expected value of r_s:

$$E(r_s \mid r_t) = r_t \cdot e^{-a \cdot (s-t)} + b \cdot \left(1 - e^{-a \cdot (s-t)}\right)$$

$$E(r_s \mid r_t) \underset{s \to +\infty}{\rightarrow} b$$

And its variance:

$$Var(r_s \mid r_t) = r_t \cdot \frac{\sigma^2}{a} \cdot \left(e^{-a \cdot (s-t)} - e^{-2 \cdot a \cdot (s-t)}\right) + b \cdot \frac{\sigma^2}{2 \cdot a} \cdot \left(1 - e^{-a \cdot (s-t)}\right)^2$$

$$Var(r_s \mid r_t) \underset{s \to +\infty}{\rightarrow} b \cdot \frac{\sigma^2}{2 \cdot a}$$

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I_q is the modified Bessel function of the first kind of order q.

The distribution function is a non-central chi-squared with 2q+2 degrees of freedom and non-centrality parameter 2u.

The asymptotic distribution function of r_s when s becomes large enough is a gamma distribution with the following density:

$$f(r) = \frac{\omega^\nu}{\Gamma(\nu)} \cdot r^{\nu-1} \cdot e^{-\omega \cdot r}$$

Where:

$$\omega = \frac{2 \cdot a}{\sigma^2}$$

$$\nu = \frac{2 \cdot a \cdot b}{\sigma^2}$$

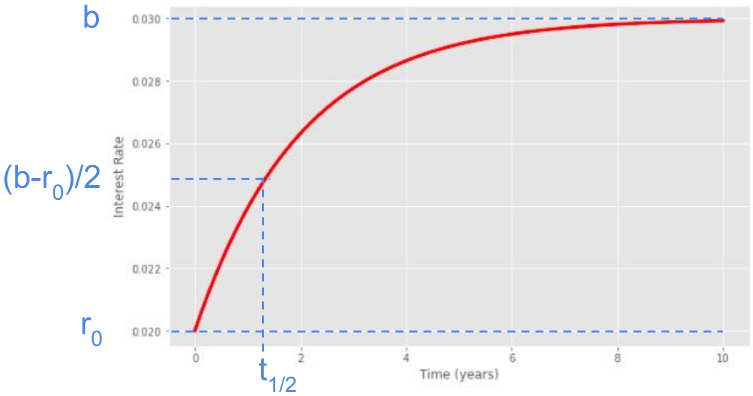
Reference: Cox, J.C., Ingersoll, J.E. and Ross, S.A. (1985), "A Theory of the Term Structure of Interest Rates", Econometrica.

Half-Life of the Mean-Reversion

It is the average time it take to be half-way back to the mean.

$$r_0 \cdot e^{-a \cdot t_{\frac{1}{2}}} + b \cdot \left(1 - e^{-a \cdot t_{\frac{1}{2}}}\right) - r_0 = \frac{b - r_0}{2}$$

$$\Rightarrow \quad t_{\frac{1}{2}} = \frac{\ln(2)}{a}$$



r_0 = 2%, a = 0.5 , b = 3%

The higher the speed of reversion the smaller the half-life.

Simulated Interest Rates Paths

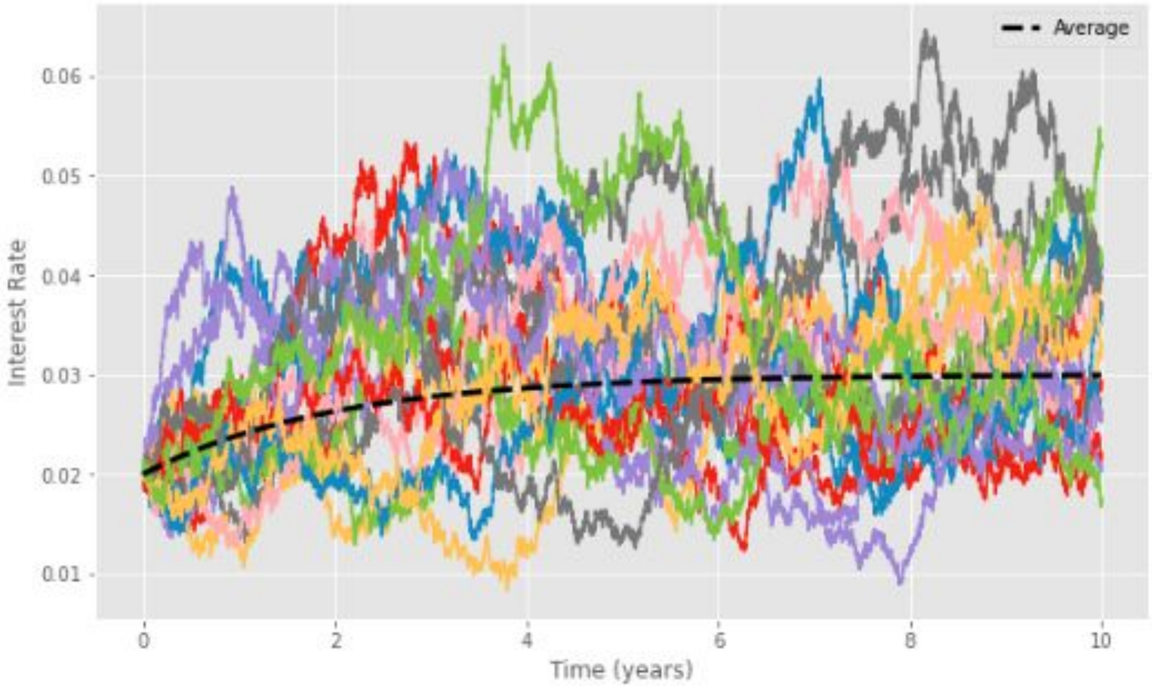
r_0 = 2%

a = 0.5

b = 3%

σ = 5%

T = 10y



Simulated Interest Rates Paths

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Zero-Coupon Bond Pricing

The price of a zero-coupon bond at the date t with maturity T has the following expression in the CIR model under the no-arbitrage assumption:

$$P(t,T) = A(t,T) \cdot e^{-B(t,T) \cdot r(t)}$$

$$A(t,T) = \left(\frac{2 \cdot \gamma \cdot e^{(\gamma + a) \cdot \frac{T-t}{2}}}{2 \cdot \gamma + (a + \gamma) \cdot \left(e^{\gamma \cdot (T-t)} - 1 \right)} \right)^{\frac{2 \cdot a \cdot b}{\sigma^2}} \qquad B(t,T) = \frac{2 \cdot \left(e^{\gamma \cdot (T-t)} - 1 \right)}{2 \cdot \gamma + (a + \gamma) \cdot \left(e^{\gamma \cdot (T-t)} - 1 \right)}$$

$$\gamma = \sqrt{a^2 + 2 \cdot \sigma^2}$$

Reference: Cox, J.C., Ingersoll, J.E. and Ross, S.A. (1985), "A Theory of the Term Structure of Interest Rates", Econometrica.

The Heston Model

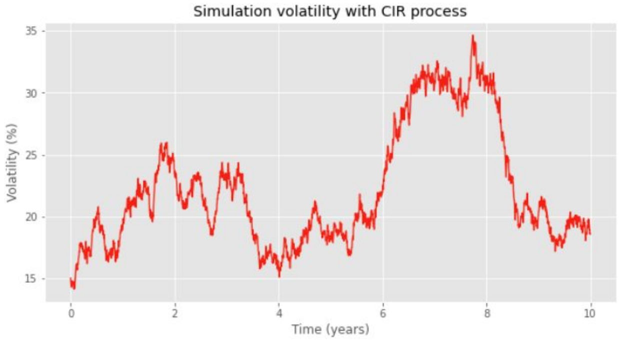
Under the "real" probability P:

$$dS_t = \mu \cdot S_t \cdot dt + \sqrt{\nu_t} \cdot S_t \cdot dW_t^{1,P}$$

$$d\nu_t = \kappa \cdot (\theta - \nu_t) \cdot dt + \xi \cdot \sqrt{\nu_t} \cdot dW_t^{2,P}$$

$$dW_t^{1,P} \cdot dW_t^{2,P} = \rho \cdot dt$$

The instantaneous variance follows a CIR process.



S_t : Underlying asset price at time t

μ : Drift of the price process

ν_t : Instantaneous variance at time t

κ : Speed of mean-reversion

θ Long-term mean of the variance

ξ : Volatility of the variance (vol of vol)

$W_t^{1,P}, W_t^{2,P}$: Wiener processes under P

ρ : Correlation of the two Wiener processes

Limits

Impossibility to have negative value, which was positive to model interest rates in pre-crisis world but can be problematic in post-crisis one.

The flexibility of the model is limited, it is typically unable to reproduce all zero-coupon curve shapes observed in the market.

It is a single factor model, with a constant volatility parameter and it does not incorporate jumps.

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