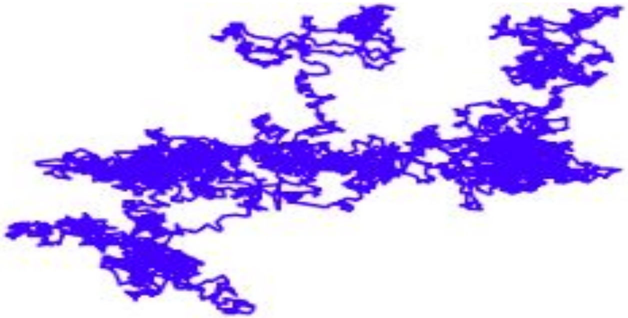
The Brownian Motion

The Brownian Motion
A bit of History

Robert Brown (1773 - 1858) a Scottish botanist discovered in 1827 the Brownian motion while observing movements of grains of pollen through a microscope.

Simulation of the Brownian motion of a particule:



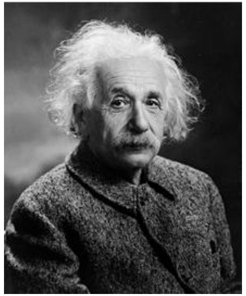
Source: Wikipedia

The Brownian Motion
A bit of History

Louis Bachelier (1870-1946) was the first person to model the stochastic process Brownian motion. He used it to model stock prices dynamic and value stock options in his thesis in 1900.

Source: Wikipedia

In 1905, Albert Einstein (1879-1955) modeled the trajectory of atoms subject to shocks, with random motion and obtained a Gaussian density.



Source: Wikipedia

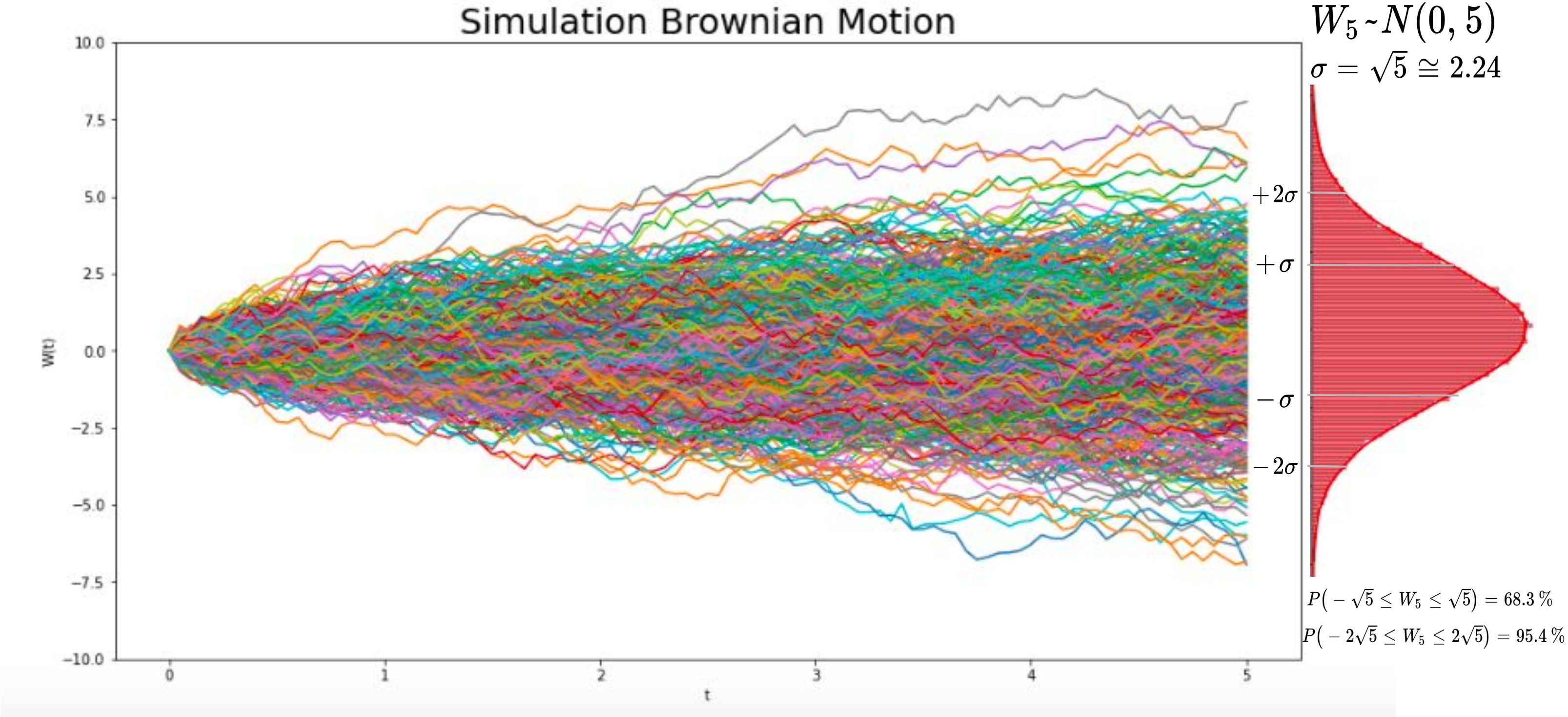
Brownian Motion

A **stochastic process** $\{W_t\}_{t\geq 0}$ is a **standard Brownian motion**, or **Wiener process** if:

- ❑ $W_0 = 0$
- ❑ It has **continuous path**
- ❑ It has **independent and stationary increments**: for every $t \geq 0, u \geq 0$

  $W_{t+u} - W_t$ are independent of past values $W_s, s \leq t$ and $W_{t+u} - W_t$ has the same distribution as W_u
- ❑ It has **Gaussian increments**: $W_{t+u} - W_t \sim N(0, u)$

Simulation of Brownian Motion



$W_5 \sim N(0, 5)$

$\sigma = \sqrt{5} \cong 2.24$

$P(-\sqrt{5} \leq W_5 \leq \sqrt{5}) = 68.3\,\%$

$P(-2\sqrt{5} \leq W_5 \leq 2\sqrt{5}) = 95.4\,\%$

Martingale

A stochastic process $\{S_t\}_{t\geq 0}$ is a **martingale** with respect to the filtration $\{F_t\}_{t\geq 0}$ and the probability P if

- $E^P(|S_t|) < \infty$
- $E^P(S_{t+s} \mid F_t) = S_t$ for $s \geq 0$

It means that the conditional expectation of S_{t+s} given all the information known at t is equal to S_t .

Martingale - Examples

1) Wiener process $\{W_t\}_{t\geq 0}$:

$$E^P(W_{t+s} \mid F_t) = E^P(W_{t+s} - W_t + W_t \mid F_t)$$
$$= E^P(W_{t+s} - W_t \mid F_t) + E^P(W_t \mid F_t)$$
$$= E^P(W_{t+s} - W_t) + W_t$$

$$\boxed{E^P(W_{t+s} \mid F_t) = W_t}$$

Martingale - Examples

2) $W_t^2 - t$:

$$\begin{aligned} E^P\left(W_{t+s}^2-(t+s) \mid F_t\right) &= E^P\left(W_{t+s}^2 \mid F_t\right)-(t+s) \\ &= E^P\left(\left(W_{t+s}-W_t+W_t\right)^2 \mid F_t\right)-(t+s) \\ &= E^P\left(\left(W_{t+s}-W_t\right)^2 \mid F_t\right)+2 \cdot E^P\left(\left(\left(W_{t+s}-W_t\right) \cdot W_t\right) \mid F_t\right)+E^P\left(W_t^2 \mid F_t\right)-(t+s) \\ &= s+2 \cdot 0+W_t^2-(t+s) \end{aligned}$$

$$E^P\left(W_{t+s}^2-(t+s) \mid W_t\right)=W_t^2-t$$

Martingale - Examples

3) $e^{\lambda \cdot W_t - \frac{1}{2}\cdot\lambda^2\cdot t}$: $E^P\left(e^{\lambda \cdot W_{t+d} - \frac{1}{2}\cdot\lambda^2\cdot(t+d)} \mid F_t\right) = e^{\lambda \cdot W_t - \frac{1}{2}\cdot\lambda^2\cdot(t+d)} \cdot E^P\left(e^{\lambda\cdot(W_{t+d}-W_t)}\right)$

$E^P\left(e^{\lambda\cdot N}\right) = e^{\frac{\lambda^2}{2}}$ when $N \sim N(0,1)$

So

$$E^P\left(e^{\lambda\cdot(W_{t+d}-W_t\}}\right) = E^P\left(e^{\lambda\cdot W_d}\right) = E^P\left(e^{\lambda\cdot\sqrt{d}\cdot W_1}\right) = e^{\frac{1}{2}\cdot\lambda^2\cdot d}$$

And we get $E^P\left(e^{\lambda \cdot W_{t+d} - \frac{1}{2}\cdot\lambda^2\cdot(t+d)} \mid F_t\right) = e^{\lambda \cdot W_t - \frac{1}{2}\cdot\lambda^2\cdot t}$

Quadratic Variation of a Brownian Motion

Let's consider $P_n[0,t] = \{t_0^n = 0 < t_1^n, \dots < t_n^n = t\}$ a partition of [0,t]

The quadratic variation of the Brownian Motion W_t is the limit of the sum of the squared changes on the partition $P_n[0,t]$ when n goes to infinity:

$$QV(W_t) = \lim_{n \to +\infty} Q_n(W_t) \text{ with } Q_n(W_t) = \sum_{i=1}^{n} \left(W_{t_i^n} - W_{t_{i-1}^n}\right)^2$$

The quadratic variation of the Brownian Motion is equal to t with probability 1

Quadratic and Absolute Variation Simulation

In the following we simulate the absolute and quadratic variations with the following partition of [0,t]: $\left\{t_0^{2^n} = 0, .. t_i^{2^n} = i \cdot \frac{t}{2^n}, ..., t_{2^n}^{2^n} = t\right\}$ by increasing n.

Absolute Variation:

$$AV(n,t) = \sum_{i=1}^{2^n} \left| W_{\frac{i \cdot t}{2^n}} - W_{\frac{(i-1) \cdot t}{2^n}} \right|$$

$$\lim_{n \to +\infty} AV(n,t) = +\infty$$

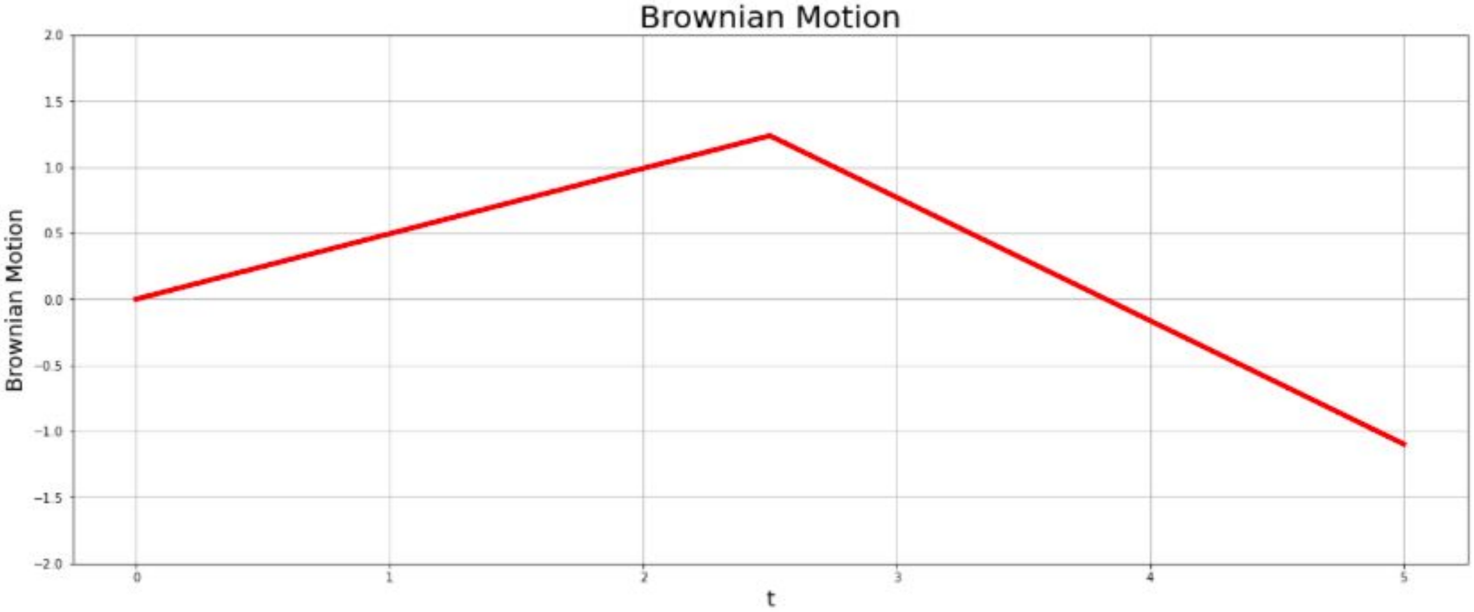
Quadratic Variation:

$$QV(n,t) = \sum_{i=1}^{2^n} \left(W_{\frac{i \cdot t}{2^n}} - W_{\frac{(i-1) \cdot t}{2^n}} \right)^2$$

$$\lim_{n \to +\infty} QV(n,t) = t$$

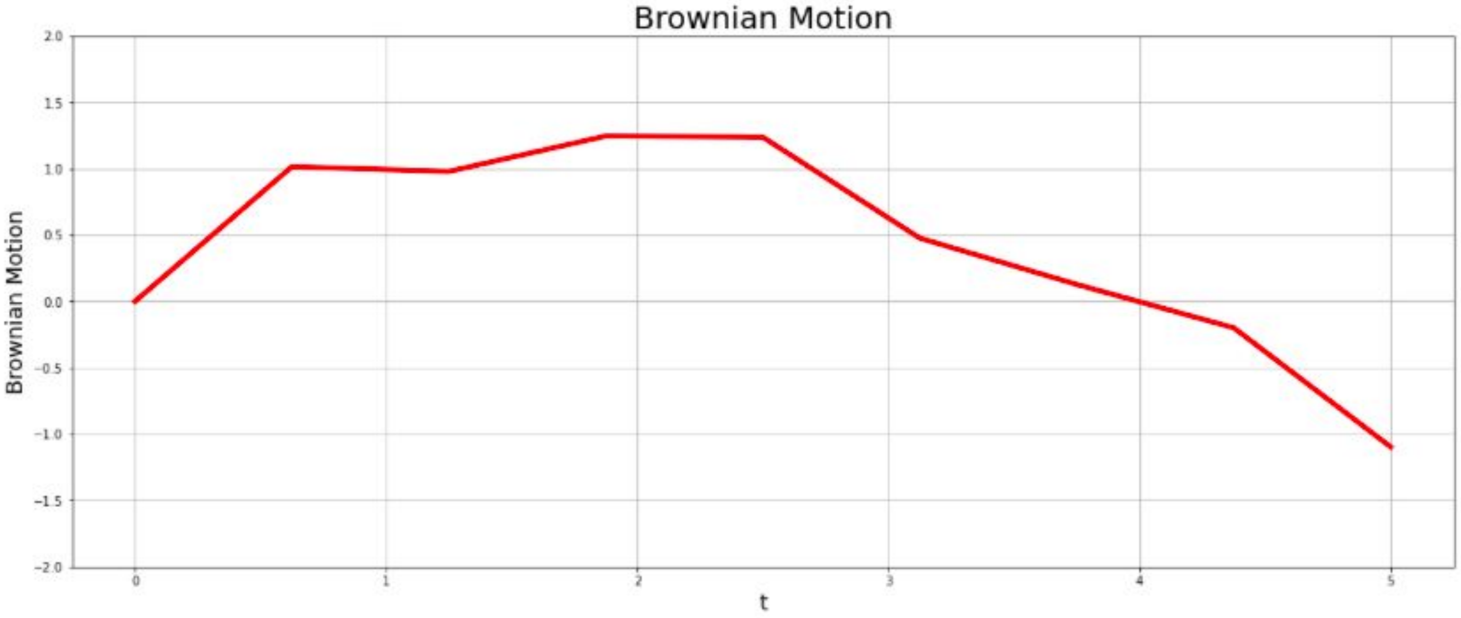
Quadratic and Absolute Variation Simulation

n=1
Absolute Variation = 4.6655
Quadratic Variation = 6.9777



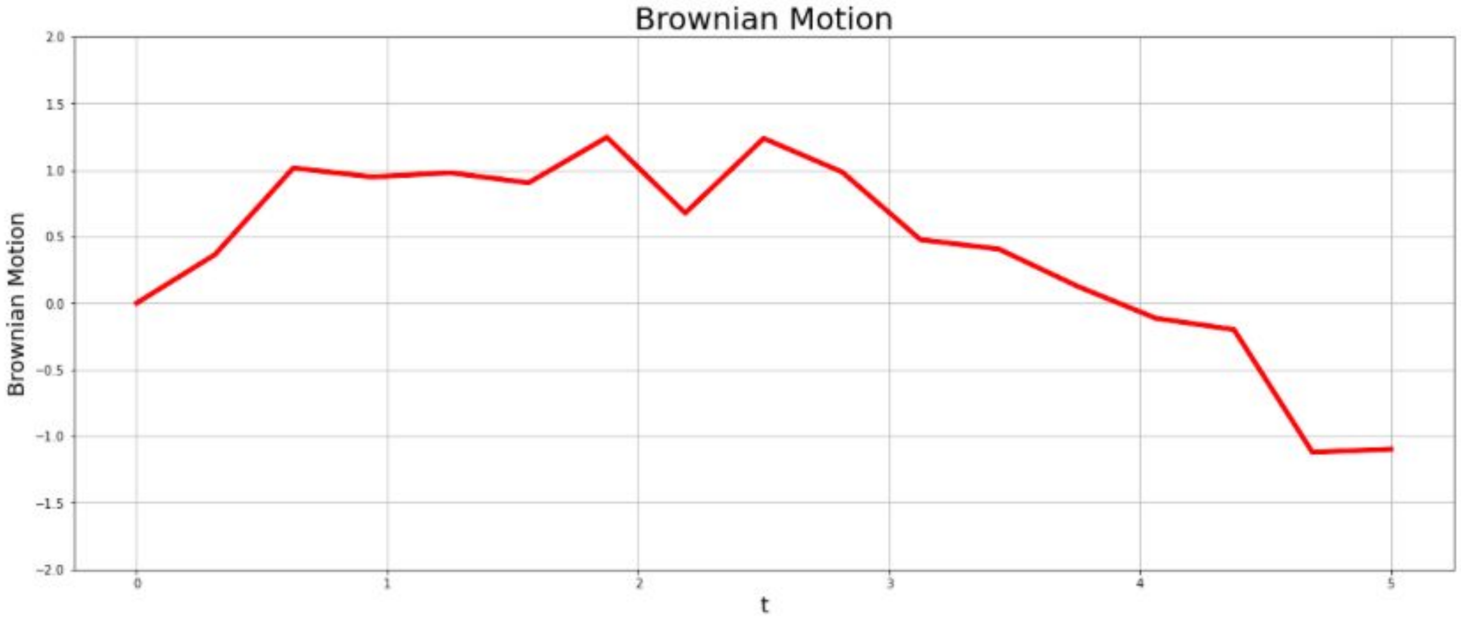
Quadratic and Absolute Variation Simulation

n=3
Absolute Variation = 4.7701
Quadratic Variation = 2.7164



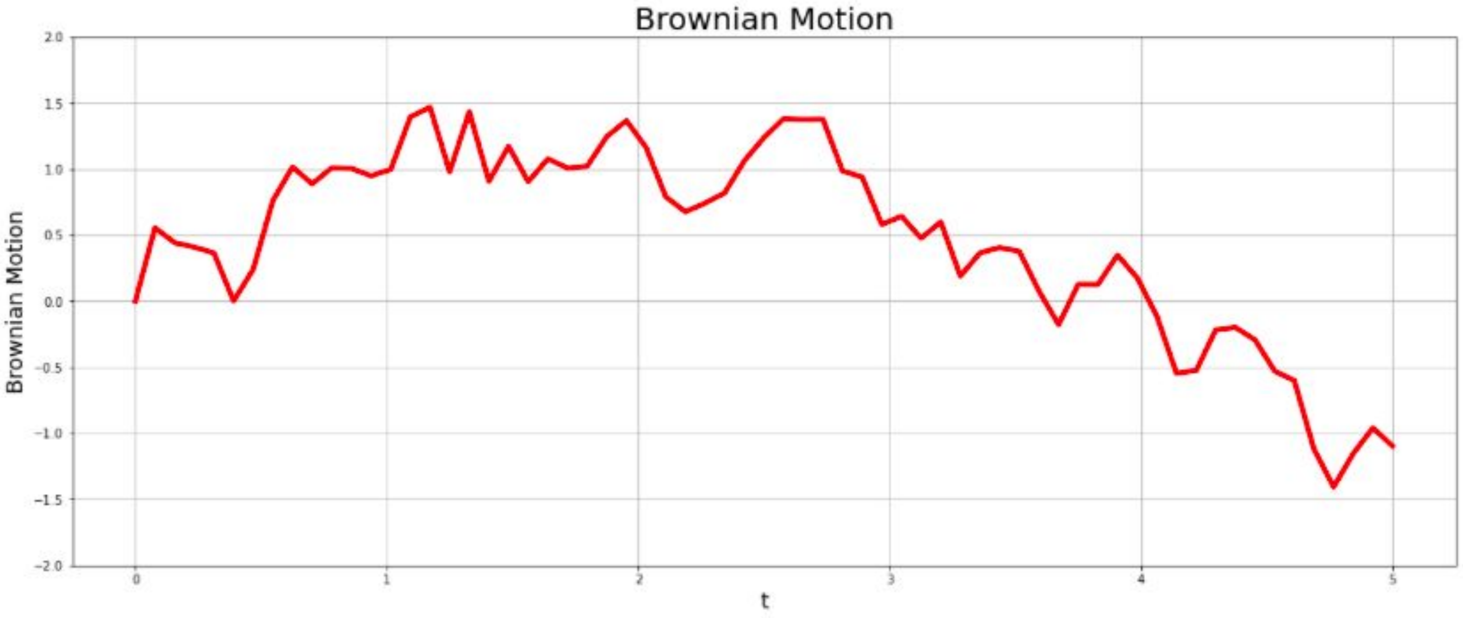
Quadratic and Absolute Variation Simulation

n=4
Absolute Variation = 6.1477
Quadratic Variation = 2.6431



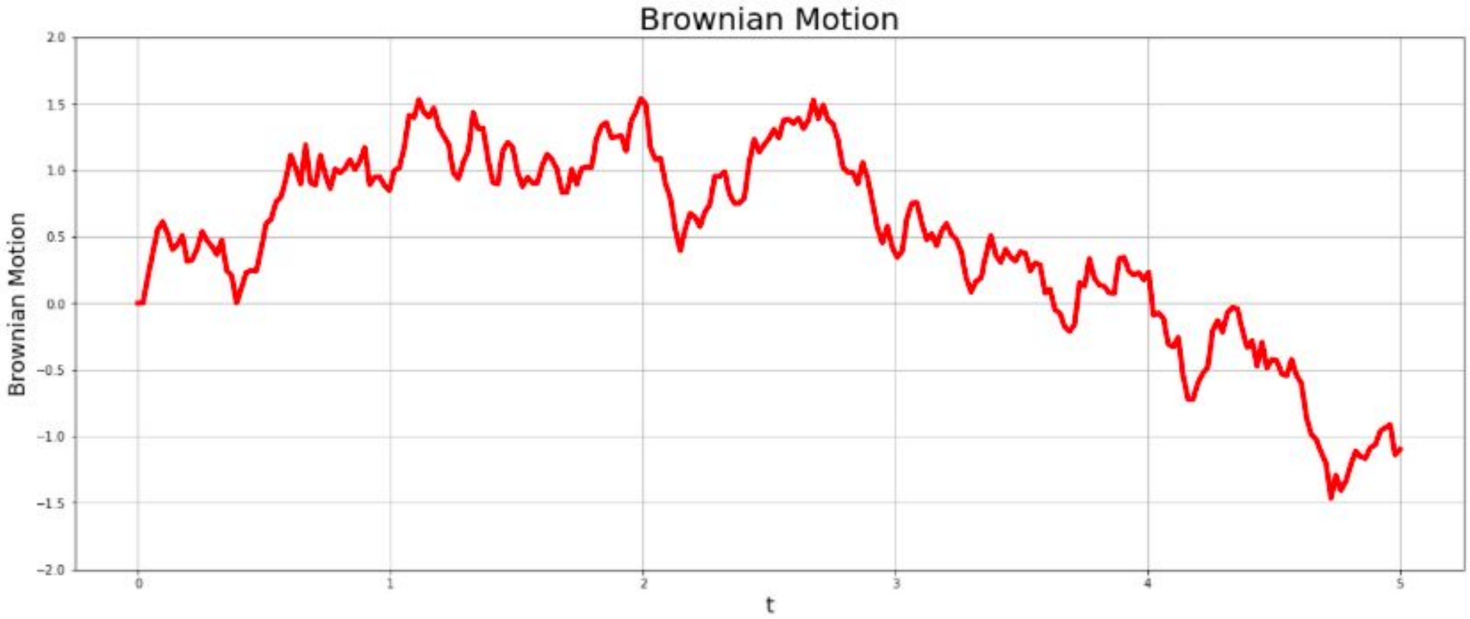
Quadratic and Absolute Variation Simulation

n=6
Absolute Variation = 13.9865
Quadratic Variation = 4.0951



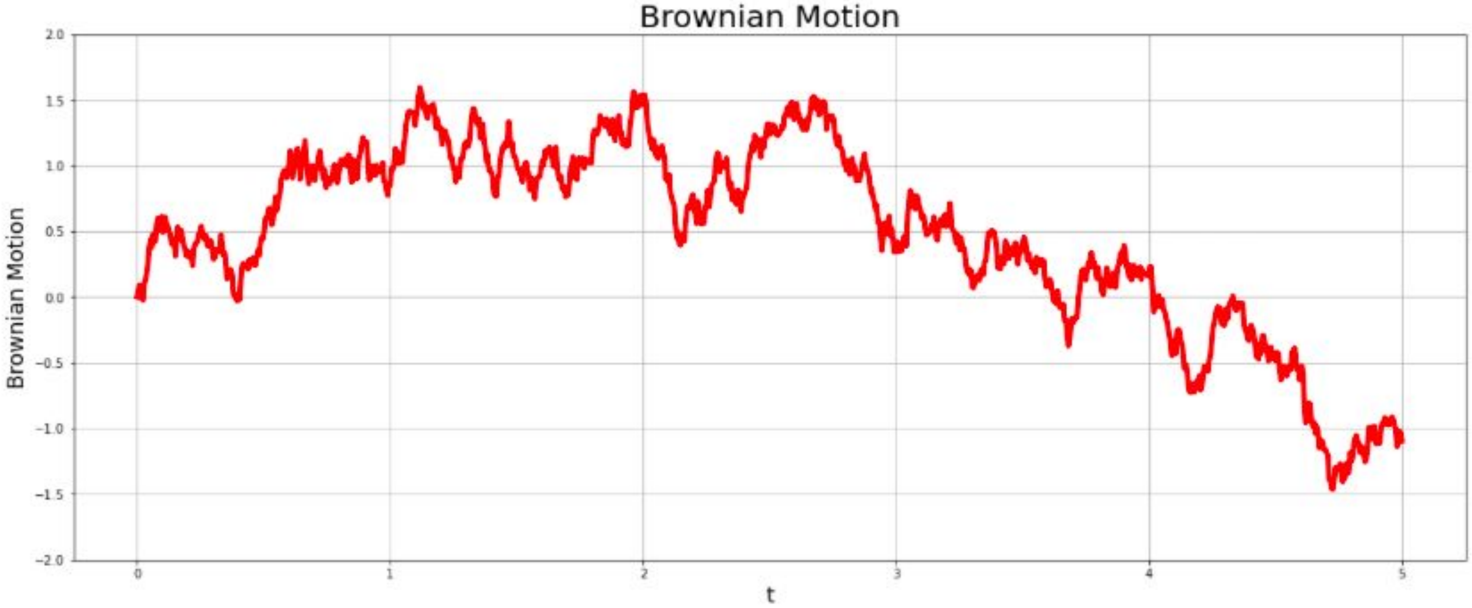
Quadratic and Absolute Variation Simulation

n=8
Absolute Variation = 27.2652
Quadratic Variation = 4.1485



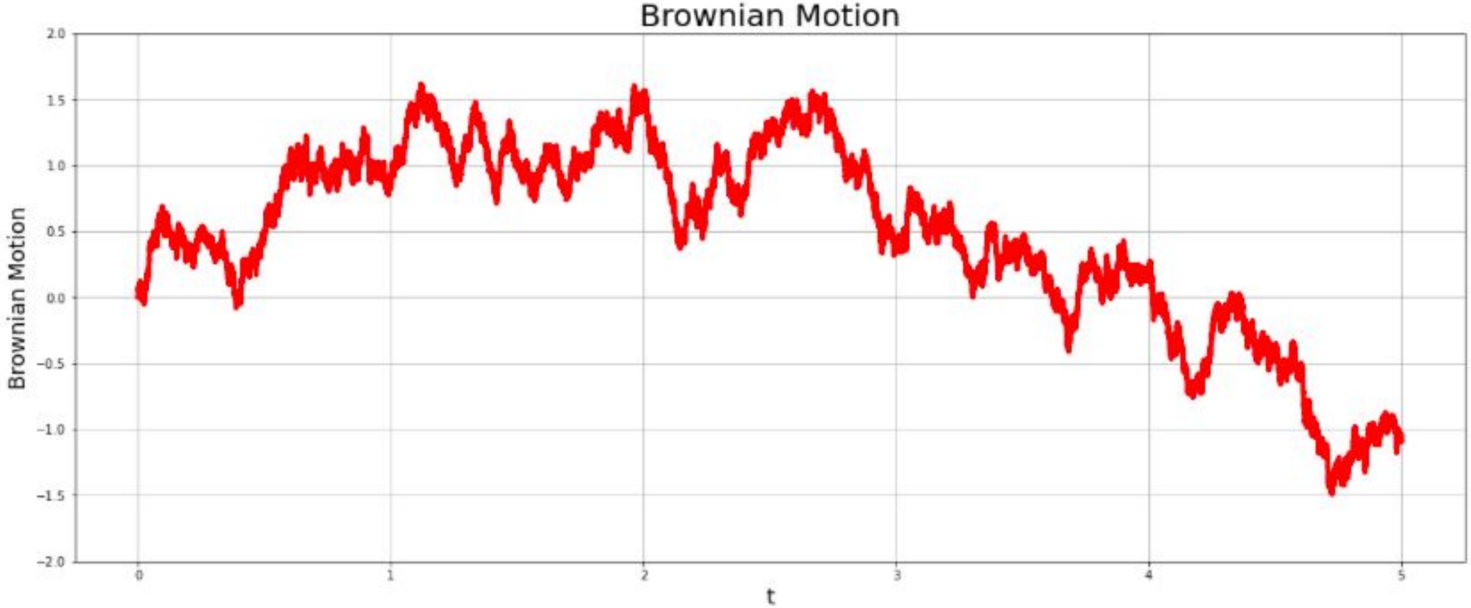
Quadratic and Absolute Variation Simulation

n=10
Absolute Variation = 57.6022
Quadratic Variation = 4.7746



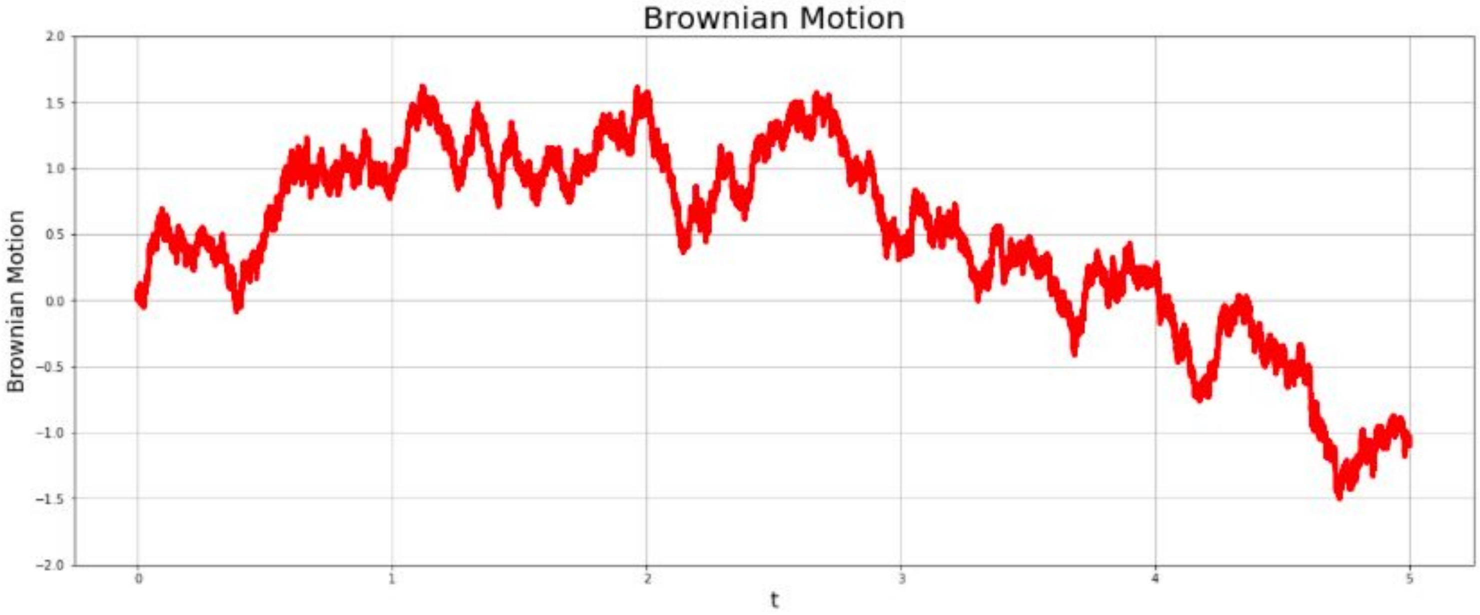
Quadratic and Absolute Variation Simulation

n=15
Absolute Variation = 322.7447
Quadratic Variation = 4.9574

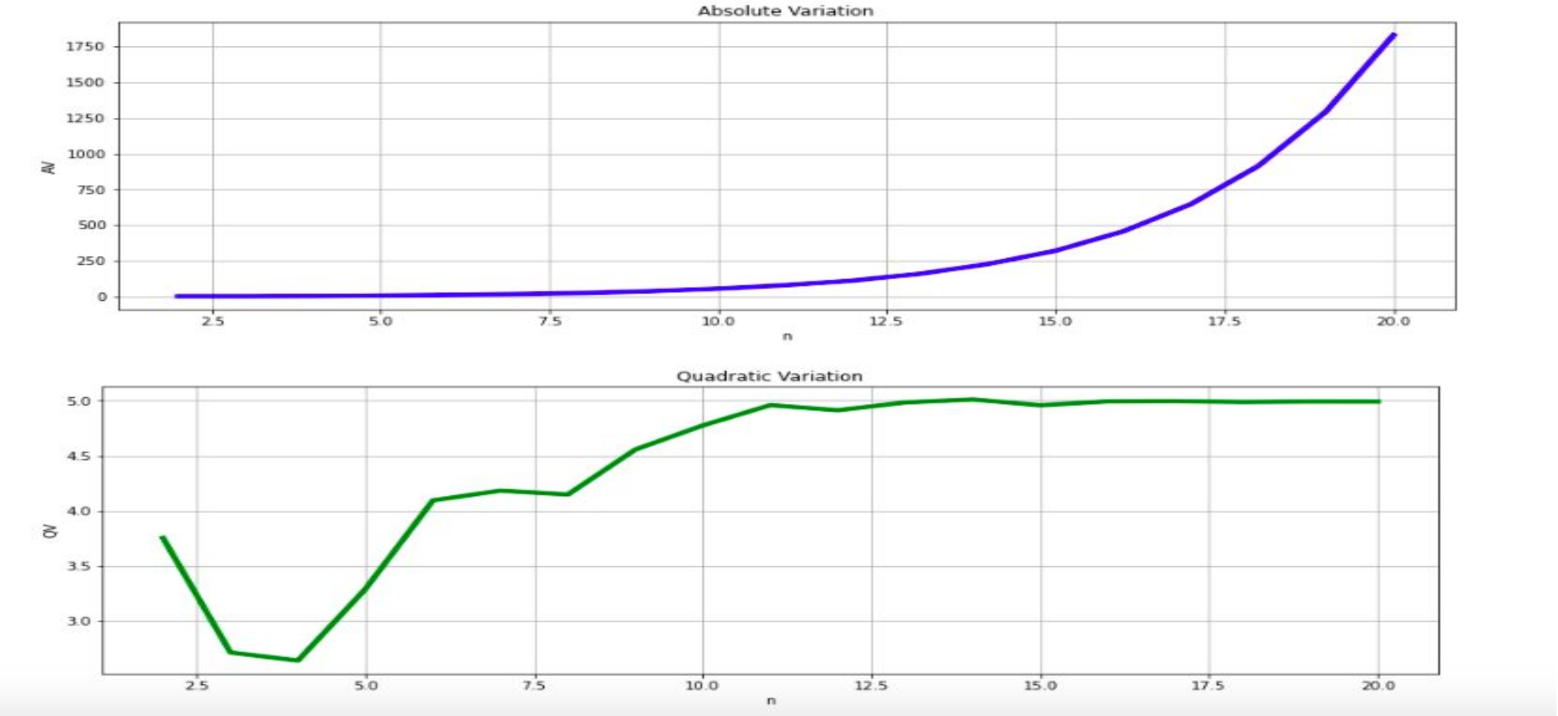


Quadratic and Absolute Variation Simulation

n=20
Absolute Variation = 1827.3135
Quadratic Variation = 4.9909



Quadratic and Absolute Variation Simulation



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