Introduction
to
Finite Difference Methods
for Option Pricing

Finite Difference Methods

The price of a European option P(S, t) is solution of the Black-Scholes PDE:

$$\frac{\partial P}{\partial t} + r \cdot \frac{\partial P}{\partial S} \cdot S + \frac{1}{2} \cdot \sigma^2 \cdot S^2 \cdot \frac{\partial^2 P}{\partial S^2} = r \cdot P$$

with the final condition, with f the payoff of the option:

$$P(S, T) = f(S_T)$$

In most cases we do not have a closed form solution of this equation.

We need to apply numerical methods to solve it.

Finite Difference Methods

Finite-difference methods (**FDM**) is the generic term for a large number of techniques that can be used for solving differential equations, approximating derivatives with finite differences.

It consists of a **discretization** of both spatial (underlying price in our case) and time intervals to obtain a finite **grid**.

We find solutions for a differential equation by **approximating** every partial derivative with **numerical methods** and we apply the solution to the points of the grid.

2-Dimensional Grid

$$S = \{0, \delta S, .. i \cdot \delta S, .., m \cdot \delta S\}$$

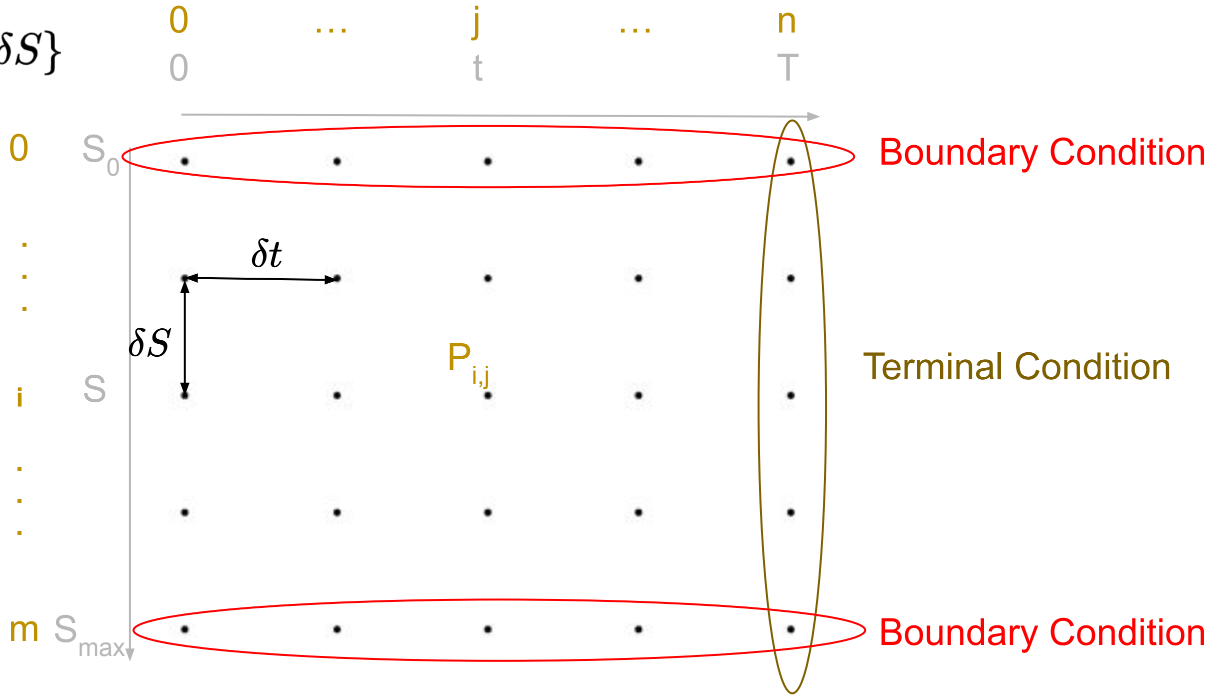
$$t = \{0, \delta t, .. j \cdot \delta t, .., n \cdot \delta t\}$$

$\delta t, \delta S$: size of discretization

$$m \cdot \delta S = S_{\max}$$

$$n \cdot \delta t = T$$

$$P_{i,j} = P(i \cdot \delta S, j \cdot \delta t)$$



Finite Difference Approximations

Let’s consider a function f continuous and n times differentiable, we derive the following **approximation** for f(x+h) from **Taylor**:

$$f(x+h) = f(x) + h \cdot f'(x) + \frac{h^2}{2} \cdot f''(x) + .. \frac{h^n}{n!} \cdot f^{(n)}(x) + O(h^{n+1})$$

Order of the error

Finite Difference Approximations
First Order

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Order of the error

$$f'(x) = \frac{f(x+h) - f(x)}{h} + O(h)$$

This is the forward difference approximation of the first derivative.

Finite Difference Approximations
First Order

Similarly we have:

$$f(x-h) = f(x) - h \cdot f'(x) + \frac{h^2}{2} \cdot f''(x) + \frac{(-1)^n}{n!} \cdot f^{(n)}(x) + O\left(h^{n+1}\right)$$

$$\Rightarrow \quad \boxed{f'(x) = \frac{f(x)-f(x-h)}{h} + O(h)}$$

This is the backward difference approximation of the first derivative.

Finite Difference Approximations
First Order

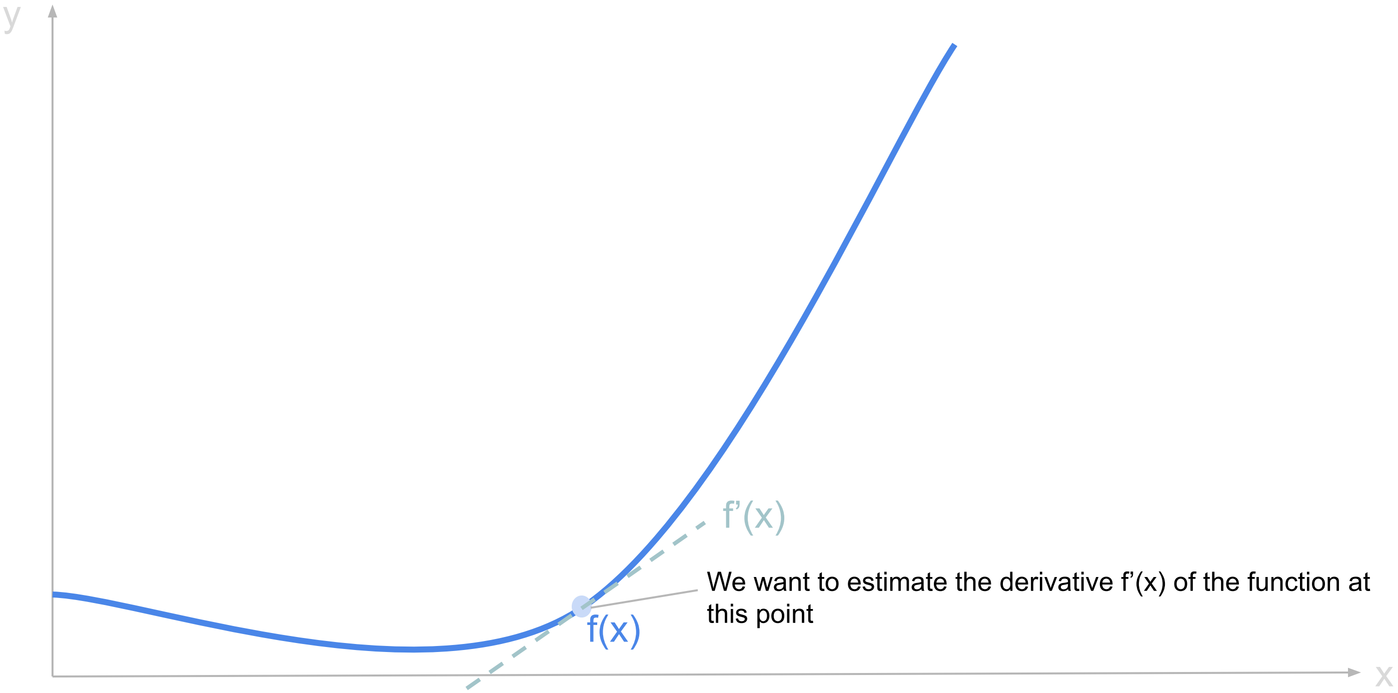
By subtracting the two developments and dividing it by 2.h we get:

$$f'(x) = \frac{f(x+h)-f(x-h)}{2\cdot h} + O\left(h^2\right)$$

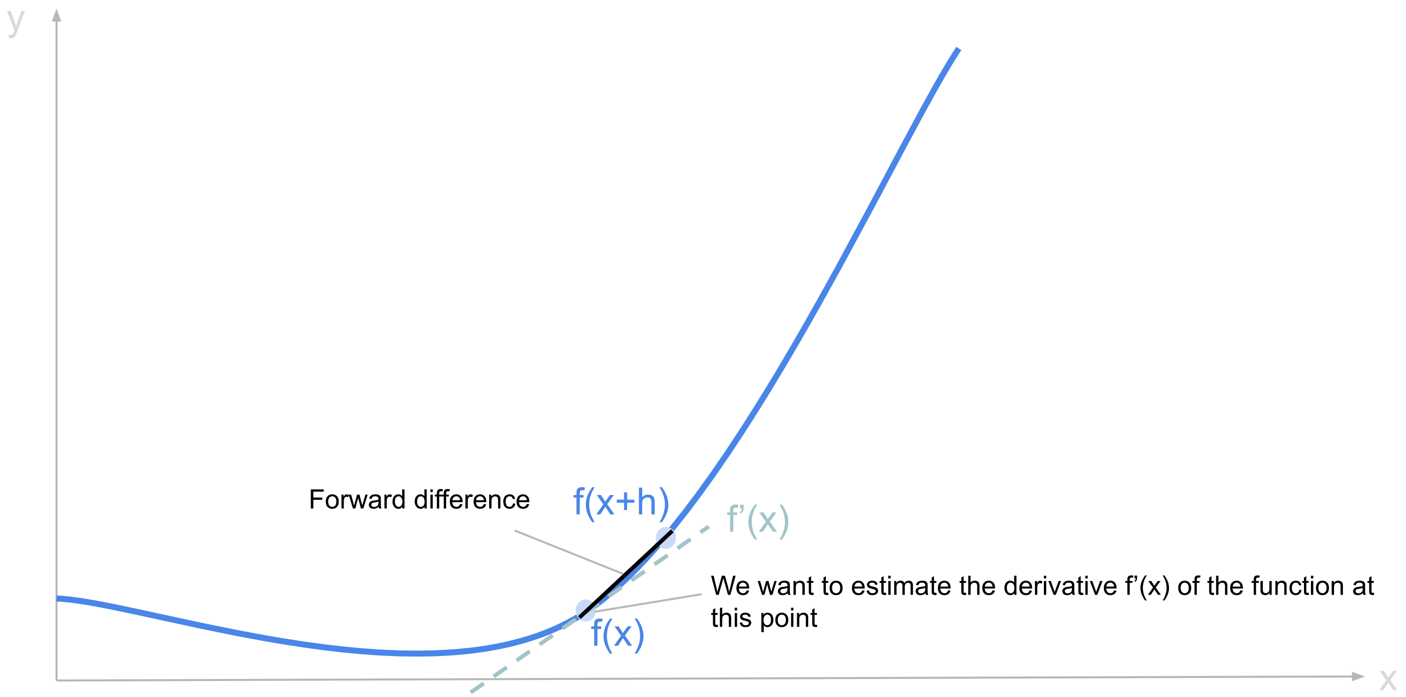
This is the central difference approximation for the first derivative.

It is the best of the three approximation with an error in h^2 vs h for the two others.

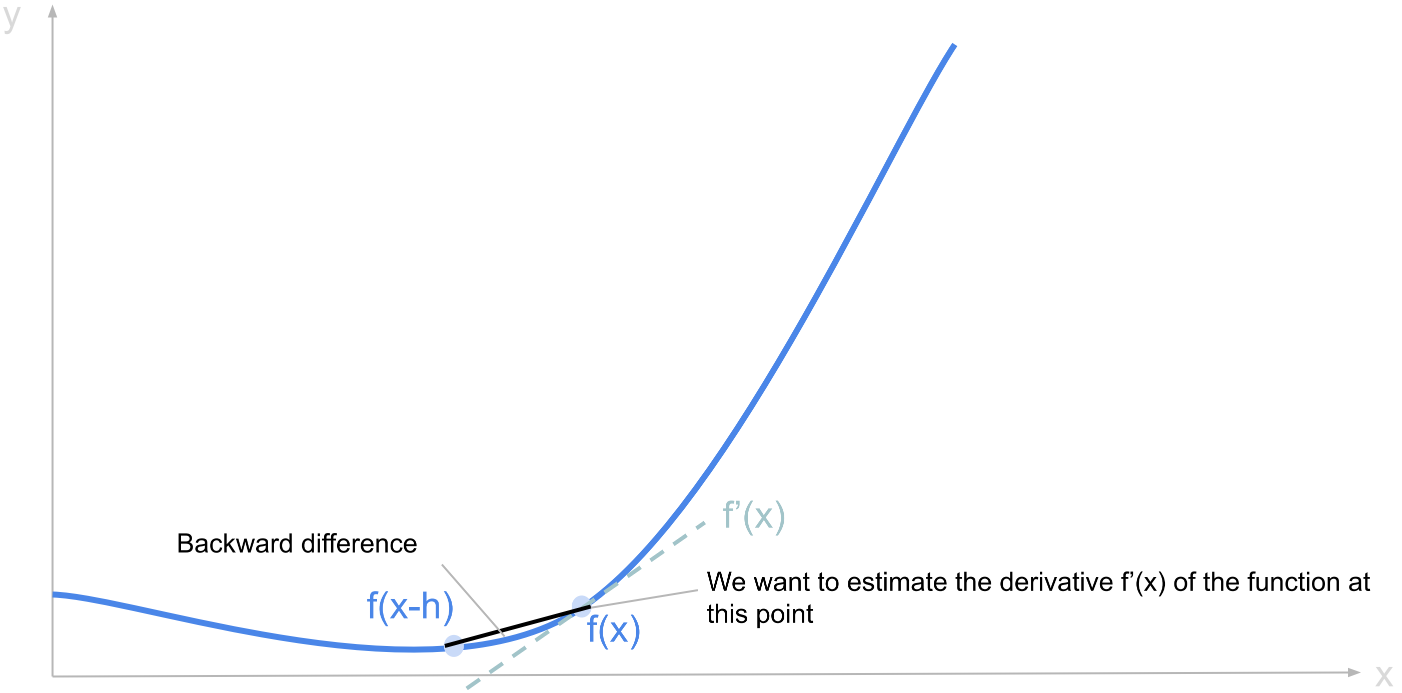
Finite Difference Approximations
First Order



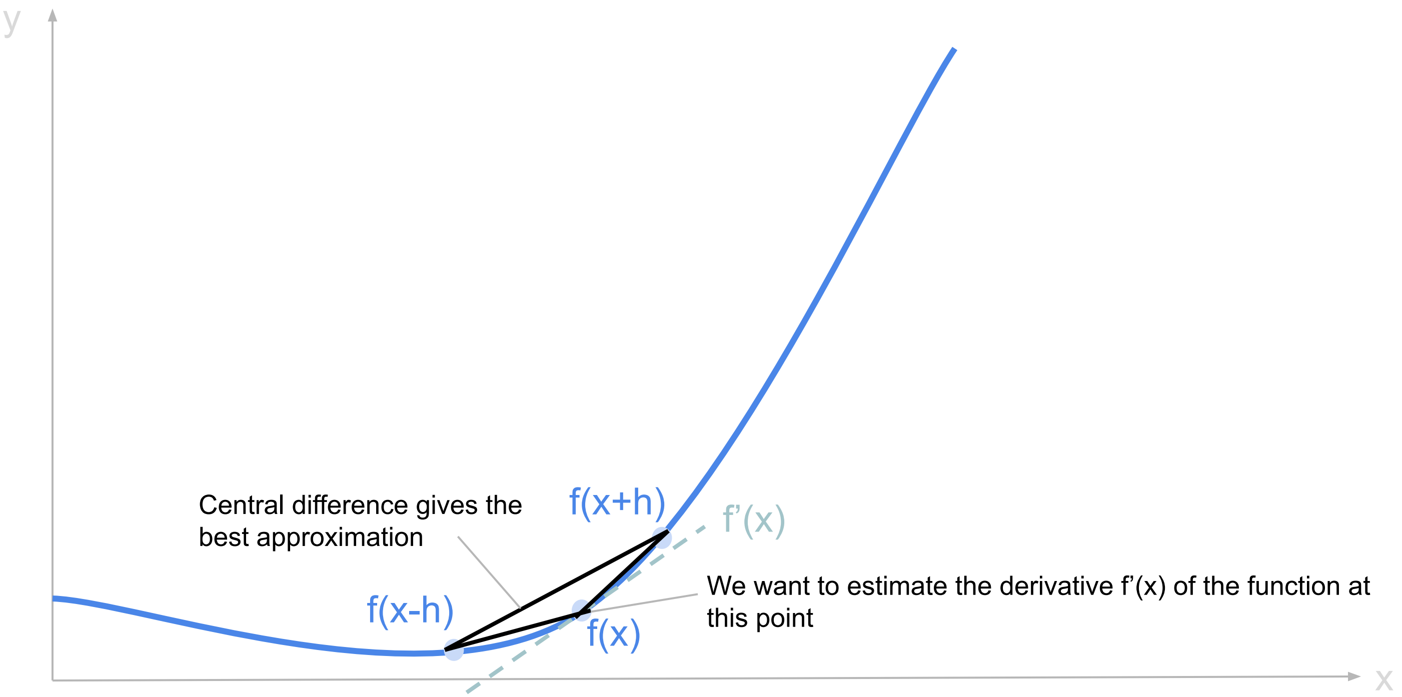
Finite Difference Approximations
First Order



Finite Difference Approximations First Order



Finite Difference Approximations First Order



Finite Difference Approximations Second Order

We also get the approximation of the second derivatives by adding the two developments and dividing by h^2.

$$f''(x) = \frac{f(x+h) - 2 \cdot f(x) + f(x-h)}{h^2} + O\left(h^2\right)$$

This is the second-order central difference approximation for the second derivative.

The error is in h^2.

Finite Difference Approximations

From the Black Scholes PDE, we need approximations for:

$$\frac{\partial P}{\partial t} + r \cdot \frac{\partial P}{\partial S} \cdot S + \frac{1}{2} \cdot \sigma^2 \cdot S^2 \cdot \frac{\partial^2 P}{\partial S^2} = r \cdot P$$

Forward Difference

$$\frac{\partial P}{\partial t} = \frac{P_{i,j+1} - P_{i,j}}{\delta t} \qquad \frac{\partial P}{\partial S} = \frac{P_{i+1,j} - P_{i,j}}{\delta S}$$

Backward Difference

$$\frac{\partial P}{\partial t} = \frac{P_{i,j} - P_{i,j-1}}{\delta t} \qquad \frac{\partial P}{\partial S} = \frac{P_{i,j} - P_{i-1,j}}{\delta S}$$

Central Difference

$$\frac{\partial P}{\partial t} = \frac{P_{i,j+1} - P_{i,j-1}}{2 \cdot \delta t} \qquad \frac{\partial P}{\partial S} = \frac{P_{i+1,j} - P_{i-1,j}}{2 \cdot \delta S}$$

Second-order Central Difference

$$\frac{\partial^2 P}{\partial S^2} = \frac{P_{i+1,j} - 2 \cdot P_{i,j} + P_{i-1,j}}{\delta S^2}$$

Using grid notations

Terminal Conditions

At its expiration, the price of an option is equal to its final payoff:

	Continuous	**Discrete**
	$P(S,T) = f(S_T)$	$P_{i,n} = f(i \cdot \delta S)$
Call Option	$P(S,T) = (S_T - K)^+$	$P_{i,n} = (i \cdot \delta S - K)^+$
Put Option	$P(S,T) = (K - S_T)^+$	$P_{i,n} = (K - i \cdot \delta S)^+$

Boundary Conditions

Dirichlet Boundary Condition (Type I)

Call Option

$$P(0,t) = 0 \qquad P(S_{\max},t) = S_{\max} - K \cdot e^{-r\cdot(T-t)}$$

$$P_{0,j} = 0 \qquad P_{m,j} = m \cdot \delta S - K \cdot e^{-r\cdot(n-j)\cdot\delta t}$$

Put Option

$$P(0,t) = K \cdot e^{-r\cdot(T-t)} \qquad P(S_{\max},t) = 0$$

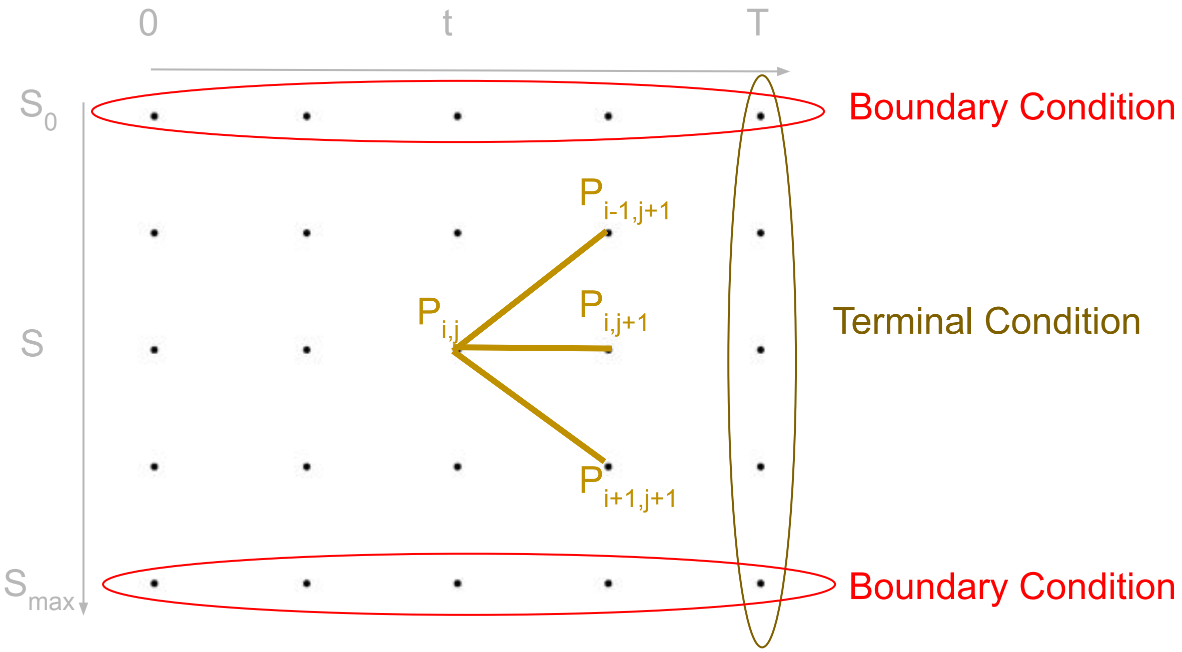
$$P_{0,j} = K \cdot e^{-r\cdot(n-j)\cdot\delta t} \qquad P_{m,j} = 0$$

Neumann Boundary Condition (Type II)

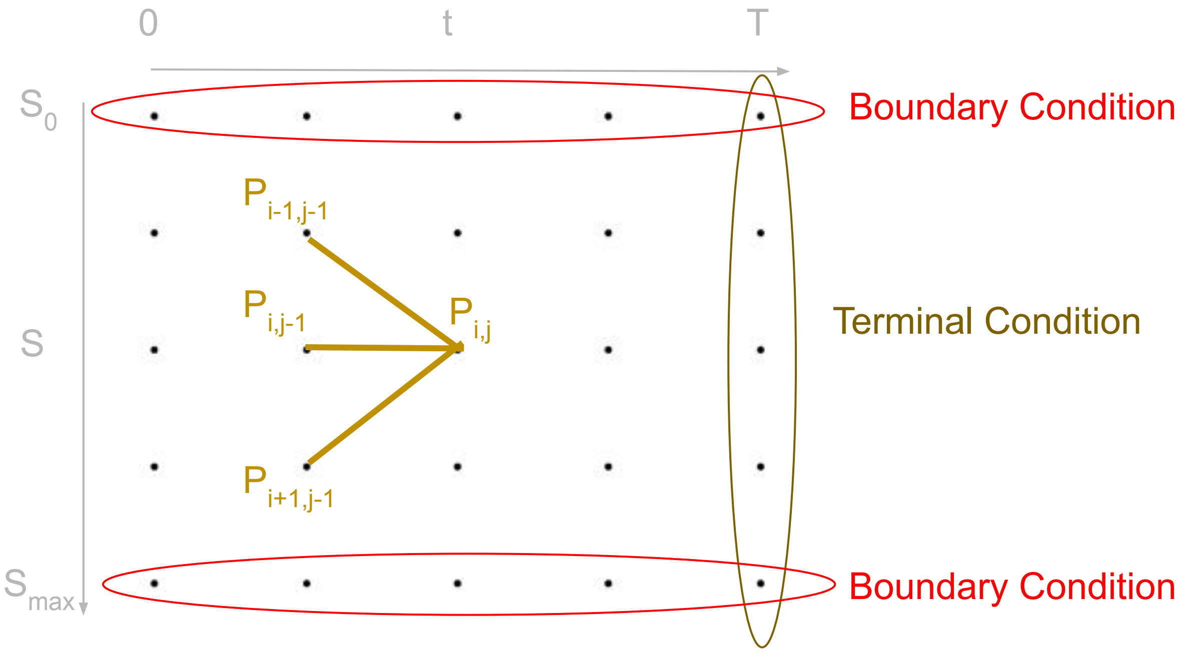
$$\frac{\partial^2 P}{\partial S^2}(\delta S, t) = 0 \qquad \frac{\partial^2 P}{\partial S^2}((m-1)\cdot\delta S, t) = 0$$

$$P_{0,j} - 2 \cdot P_{1,j} + P_{2,j} = 0 \qquad P_{m-2,j} - 2 \cdot P_{m-1,j} + P_{m,j} = 0$$

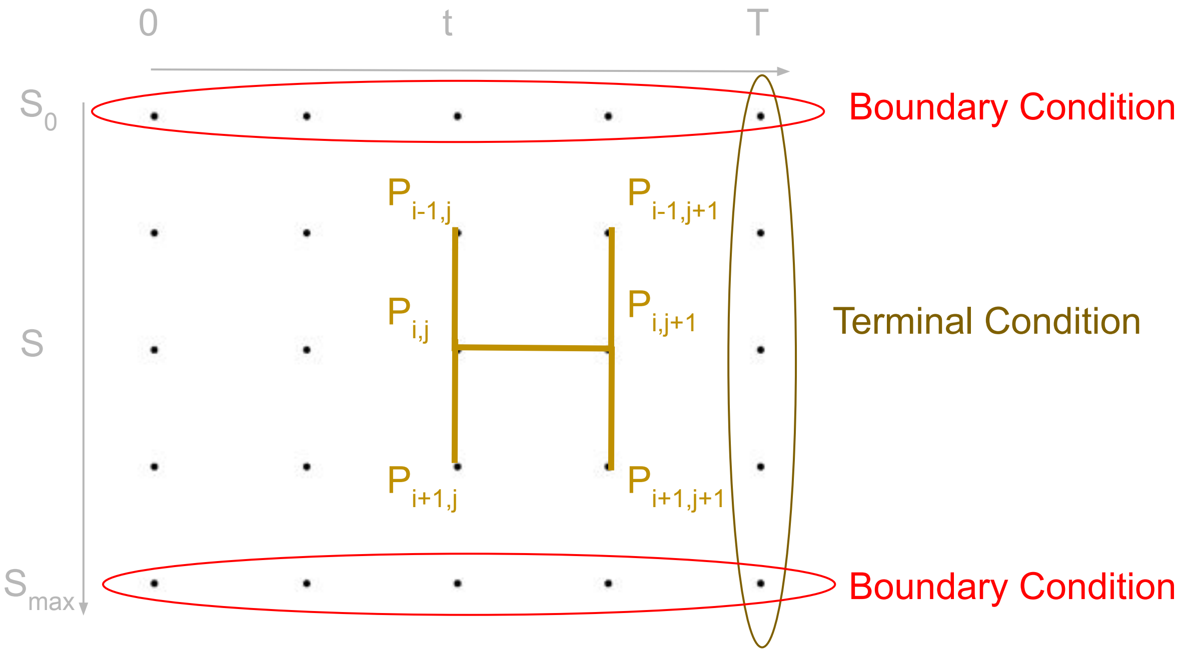
Explicit Method



Implicit Method



Crank-Nicolson Method



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