QUANT NEXT

Homogeneity of Option Prices

What Does It Tell Us?

Call Prices are Homogeneous of Order 1 in Many Classes of Models

The Option price $C(S_0, K, T)$ is **homogeneous of order 1** in (S_0, K) when

$$C(\lambda.S_0, \lambda.K, T) = \lambda.C(S_0, K, T)$$

It means that if you double the current value of the asset price S_0 and the strike price K, you also double the price of the option.

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This is true for log-type models where increments of the log of the asset price are independent of the current value of the asset price under the risk neutral probability Q[1].

This is true for a large number of models: Black-Scholes, Merton's jump diffusion, Variance Gamma, Heston, …

[1]Reference: M.S. Joshi (2001) "Log-Type Models, Homogeneity Of Option Prices And Convexity"

Call Prices are Homogeneous of Order 1 in Many Classes of Models

Typically, when you can write the asset price as following:

$$S_t = S_0 \cdot e^{X_t}$$

The price of the option is the discounted expectation of its future payoff:

$$C(S_0, K, T) = e^{-r \cdot T} \cdot E^Q\left((S_T - K)^+\right)$$
$$C(S_0, K, T) = e^{-r \cdot T} \cdot E^Q\left(\left(S_0 \cdot e^{X_T} - K\right)^+\right)$$

$$\boxed{C(\lambda \cdot S_0, \lambda \cdot K, T) = \lambda \cdot C(S_0, K, T)}$$

What Does it Tell Us?

$$\lambda \cdot C(S_0, K, T) = C(\lambda \cdot S_0, \lambda \cdot K, T)$$

If you differentiate both sides of this equation with respect to λ you get:

$$C = S_0 \cdot \frac{\partial C}{\partial S_0} + K \cdot \frac{\partial C}{\partial K}$$

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If you can write the call price in the following form:

$$C = S_0 \cdot P_1 - K \cdot e^{-r \cdot T} \cdot P_2$$

You get the two following relationships:

$$P_1 = \frac{\partial C}{\partial S_0} \qquad P_2 = -e^{r \cdot T} \cdot \frac{\partial C}{\partial K}$$

What Does it Tell Us?

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$$N(d_1) = \frac{\partial C}{\partial S_0}$$

$N(d_1)$ is equal to the Δ of the option.

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$$P_2 = N(d_2) = P(S_T \geq K)$$

$$P(S_T \geq K) = -e^{r \cdot T} \cdot \frac{\partial C}{\partial K}$$

And if we differential both sides with respect to K we obtain a **direct relationship between the risk neutral density and the second derivative of the call price with respect to the strike price. This is a special case of the Breeden-Litzenberger formula.**

$$f_{S_T}(K) = e^{r \cdot T} \cdot \frac{\partial^2 C}{\partial K^2}$$

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