Calibration of The Vasicek Model to Historical Data with Python Code

The Vasicek Model

The instantaneous interest rate r_t follows the following stochastic differential equation (SDE):

$$dr_t = \underbrace{a}_{\substack{\text{Speed of reversion} \\ a>0}} \cdot (\underbrace{b}_{\text{Long-term mean}} - r_t) \cdot dt + \underbrace{\sigma}_{\text{Instantaneous volatility}} \cdot dW_t$$

Calibration Methods

We present two methods for calibrating the Vasicek model to historical data:

1) Least Squares
2) Maximum Likelihood Estimation

Least Squares

The Euler-Maruyama discretisation method of the Vasicek model gives:

$$r_{t+\delta t} - r_t = a \cdot (b - r_t) \cdot \delta t + \sigma \cdot \sqrt{\delta t} \cdot \varepsilon \qquad \varepsilon \sim N(0,1)$$

Which can be rewritten as:

$$r_{t+\delta t} = \underbrace{(1 - a \cdot \delta t)}_{\alpha} \cdot r_t + \underbrace{a \cdot b \cdot \delta t}_{\beta} + \underbrace{\sigma \cdot \sqrt{\delta t} \cdot \varepsilon}_{\xi \text{ iid normally distributed}}$$

$$r_{t+\delta t} = \alpha \cdot r_t + \beta + \xi$$

Least Squares

The Euler-Maruyama discretisation method of the Vasicek model gives:

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ξ iid normally distributed

$$r_{t+\delta t} = \alpha \cdot r_t + \beta + \xi$$

$\left(\widehat{\alpha}, \widehat{\beta}\right)$ estimated by least squares method (linear regression of $r_{t+\delta t}$ on r_t)

$$\left(\widehat{a}, \hat{b}\right) = \left(\frac{1-\widehat{\alpha}}{\delta t}, \frac{\widehat{\beta}}{1-\widehat{\alpha}}\right) \qquad \widehat{\sigma} = \sqrt{\frac{Var(\xi)}{\delta t}}$$

Maximum Likelihood Estimation

Knowing r_t, the solution of the SDE $r_{t+\delta t}$ is equal to:

$$r_{t+\delta t} = r_t \cdot e^{-a \cdot \delta t} + b \cdot \left(1 - e^{-a \cdot \delta t}\right) + \sigma \cdot e^{-a \cdot \delta t} \cdot \int_t^{t+\delta t} e^{a \cdot s} \cdot dW_s$$

Conditionally to r_t, $r_{t+\delta t}$ follows a normal distribution.

$$r_{t+\delta t} \sim N\left(r_t \cdot e^{-a \cdot \delta t} + \underbrace{b \cdot \left(1 - e^{-a \cdot \delta t}\right)}_{\beta}, \underbrace{\frac{\sigma^2}{2 \cdot a} \cdot \left(1 - e^{-2 \cdot a \cdot \delta t}\right)}_{\Sigma^2}\right)$$

The probability distribution function is:

$$f(r_t, t; r_{t+\delta t}, t + \delta t) = \frac{1}{\sqrt{2 \cdot \pi \cdot \Sigma^2}} \cdot e^{-\frac{1}{2} \cdot \frac{\left(r_{t+\delta t} - r_t \cdot e^{-a \cdot \delta t} - \beta\right)^2}{\Sigma^2}}$$

Maximum Likelihood Estimation

The log-likelihood on partition of time $[t_0, t_1, \ldots, t_i, \ldots, t_n]$ with $t_i - t_{i-1} = \delta t$ is:

$$\ln\left(\prod_{i=0}^{n-1} f(r_{t_i}, t_i; r_{t_{i+1}}, t_{i+1})\right)$$

You can easily check it has the following expression

$$L(a, b, \sigma) = -\frac{n}{2} \cdot \ln\left(\frac{\sigma^2}{2 \cdot a} \cdot \left(1 - e^{-2 \cdot a \cdot \delta t}\right)\right) - \frac{n}{2} \cdot \ln(2 \cdot \pi)$$
$$- \frac{a}{\sigma^2 \cdot \left(1 - e^{-2 \cdot a \cdot \delta t}\right)} \cdot \sum_{i=0}^{n-1} \left(r_{t_{i+1}} - r_{t_i} \cdot e^{-a \cdot \delta t} - b \cdot \left(1 - e^{-a \cdot \delta t}\right)\right)^2$$

Maximum Likelihood Estimation

a, b, σ maximizing the log likelihood have the following expressions:

$$\widehat{a} = \frac{1}{\delta t} \cdot \ln\left(\frac{S_0 - \hat{b}}{S_1 - \hat{b}}\right), \quad \hat{b} = \frac{S_1 \cdot S_{00} - S_0 \cdot S_{01}}{S_0 \cdot S_1 - S_0^2 - S_{01} + S_{00}}, \quad \widehat{\sigma^2} = \frac{1}{n \cdot \beta \cdot \left(1 - \frac{1}{2} \cdot a \cdot \beta\right)} \cdot \sum_{i=1}^{n} \left(r_{t_i} - m_{t_{i-1}}(t_i)\right)^2$$

where:

$$S_0 = \frac{1}{n} \cdot \sum_{i=1}^{n} r_{t_{i-1}}, \quad S_1 = \frac{1}{n} \cdot \sum_{i=1}^{n} r_{t_i}$$

$$S_{00} = \frac{1}{n} \cdot \sum_{i=1}^{n} r_{t_{i-1}} \cdot r_{t_{i-1}}, \quad S_{01} = \frac{1}{n} \cdot \sum_{i=1}^{n} r_{t_{i-1}} \cdot r_{t_i}$$

$$\beta = \frac{1}{a} \cdot \left(1 - e^{-a \cdot \delta t}\right), \quad m_{t_{i-1}}(t_i) = b \cdot a \cdot \beta + r_{t_{i-1}} \cdot (1 - a \cdot \beta))$$

Reference: Fergusson, K., Platen, E. (2015), "Application of maximum likelihood estimation to stochastic short rate models”

Python Code

Import Libraries

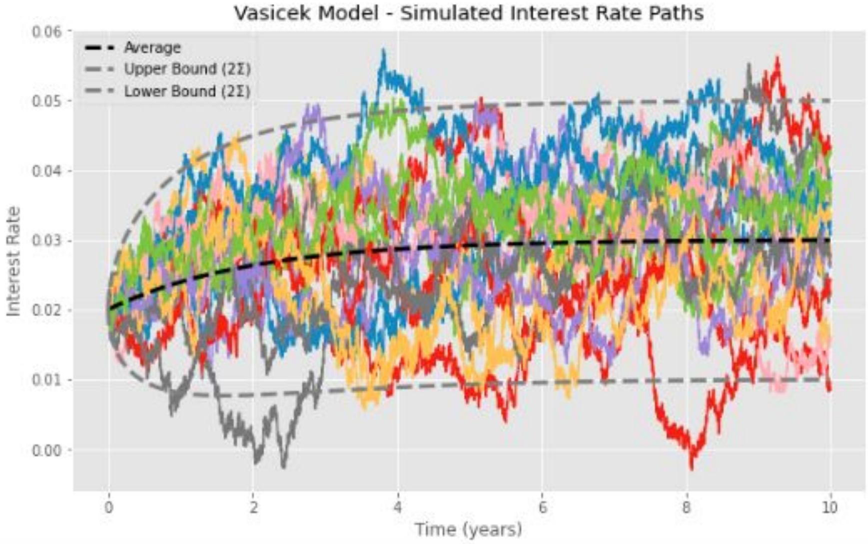
```
import matplotlib.pyplot as plt
plt.style.use('ggplot')
import math
import numpy as np
import pandas as pd
from scipy.stats import norm
%matplotlib inline
from sklearn.linear_model import LinearRegression
```

Simulation Vasicek Process with Euler-Maruyama Discretisation

```
def vasicek(r0, a, b, sigma, T, num_steps, num_paths):
    dt = T / num_steps
    rates = np.zeros((num_steps + 1, num_paths))
    rates[0] = r0

    for t in range(1, num_steps + 1):
        dW = np.random.normal(0, 1, num_paths)
        rates[t] = rates[t - 1] + a * (b - rates[t - 1]) * dt + sigma * np.sqrt(dt) * dW

    return rates
```

Python Code

```python
# Model parameters
r0 = 0.02  # Initial short rate
a = 0.5    # Mean reversion speed
b = 0.03  # Long-term mean
sigma = 0.01    # Volatility
T = 10       # Time horizon
num_steps = 10000  # Number of steps
num_paths = 20   # Number of paths

# Simulate Vasicek model
simulated_rates = vasicek(r0, a, b, sigma, T, num_steps, num_paths)

#average_rates = np.mean(simulated_rates, axis=1)

# Time axis
time_axis = np.linspace(0, T, num_steps + 1)

#average value
average_rates = [r0 * np.exp(-a * t) + b * (1 - np.exp(-a * t)) for t in time_axis]

# standard deviation
std_dev = [(sigma**2 / (2 * a) * (1 - np.exp(-2 * a * t)))**.5 for t in time_axis]

# Calculate upper and lower bounds (±2 sigma)
upper_bound = [average_rates[i] + 2 * std_dev[i] for i in range(len(time_axis))]
lower_bound = [average_rates[i] - 2 * std_dev[i] for i in range(len(time_axis))]

# Plotting multiple paths with time on x-axis
plt.figure(figsize=(10, 6))
plt.title('Vasicek Model - Simulated Interest Rate Paths')
plt.xlabel('Time (years)')
plt.ylabel('Interest Rate')
for i in range(num_paths):
    plt.plot(time_axis, simulated_rates[:, i])

plt.plot(time_axis, average_rates, color='black',linestyle='--', label ='Average', linewidth = 3)
plt.plot(time_axis, upper_bound, color='grey', linestyle='--', label='Upper Bound (2Σ)', linewidth = 3)
plt.plot(time_axis, lower_bound, color='grey', linestyle='--', label='Lower Bound (2Σ)', linewidth = 3)
plt.legend()
plt.show()
```



Python Code

Real parameters, unknown, that we want to estimate:

a = 0.5, b = 0.03, sigma = 0.01

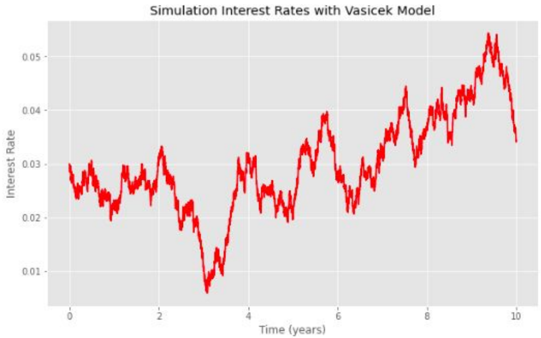
We simulate one path, with these unknown parameters, this is our historical dataset:

Real Parameters

```
# Model parameters we want to estimate
a = .5    # Mean reversion speed
b = 0.03  # Long-term mean
sigma = 0.01    # Volatility
```

Simulation one path

```
r0 = 0.03   # Initial short rate
T = 10      # Time horizon
num_steps = 10000   # Number of steps
num_paths = 1    # Number of paths

# Simulate Vasicek model
simulated_rates = vasicek(r0, a, b, sigma, T, num_steps, num_paths)

# Time axis
time_axis = np.linspace(0, T, num_steps + 1)

# Plotting multiple paths with time on x-axis
plt.figure(figsize=(10, 6))
plt.title('Simulation Interest Rates with Vasicek Model')
plt.xlabel('Time (years)')
plt.ylabel('Interest Rate')
for i in range(num_paths):
    plt.plot(time_axis, simulated_rates[:, i], color='red')

plt.show()
```



Python Code

Real parameters, unknown, that we want to estimate:

a = 0.5, b = 0.03, sigma = 0.01

Least Squares

```
def Vasicek_LS(r, dt):

    #Linear Regression
    r0 = r[:-1,]
    r1 = r[1:, 0]
    reg = LinearRegression().fit(r0, r1)

    #estimation a and b
    a_LS = (1 - reg.coef_) / dt
    b_LS = reg.intercept_ / dt / a_LS

    #estimation sigma
    epsilon = r[1:, 0] - r[:-1,0] * reg.coef_
    sigma_LS = np.std(epsilon) / dt**.5

    return a_LS[0], b_LS[0], sigma_LS
```

```
LS_Estimate = Vasicek_LS(simulated_rates, T / num_steps)
print("a_est: " + str(np.round(LS_Estimate[0],3)))
print("b_est: " + str(np.round(LS_Estimate[1],3)))
print("sigma_est: " + str(np.round(LS_Estimate[2],3)))
```

```
a_est: 0.551
b_est: 0.031
sigma_est: 0.01
```

Maximum Likelihood Estimation

```
def Vasicek_MLE(r, dt):
    r = r[:, 0]
    n = len(r)
    #estimation a and b
    S0 = 0
    S1 = 0
    S00 = 0
    S01 = 0
    for i in range(n-1):
        S0 = S0 + r[i]
        S1 = S1 + r[i + 1]
        S00 = S00 + r[i] * r[i]
        S01 = S01 + r[i] * r[i + 1]
    S0 = S0 / (n-1)
    S1 = S1 / (n-1)
    S00 = S00 / (n-1)
    S01 = S01 / (n-1)
    b_MLE = (S1 * S00 - S0 * S01) / (S0 * S1 - S0**2 - S01 + S00)
    a_MLE = 1 / dt * np.log((S0 - b_MLE) / (S1 - b_MLE))

    #estimation sigma
    beta = 1 / a * (1 - np.exp(-a * dt))
    temp = 0
    for i in range(n-1):
        mi = b * a * beta + r[i] * (1 - a * beta)
        temp = temp + (r[i+1] - mi)**2
    sigma_MLE = (1 / ((n - 1) * beta * (1 - .5 * a * beta)) * temp)**.5
    return a_MLE, b_MLE, sigma_MLE
```

```
MLE_Estimate = Vasicek_MLE(simulated_rates, T / num_steps)
print("a_est: " + str(np.round(MLE_Estimate[0],3)))
print("b_est: " + str(np.round(MLE_Estimate[1],3)))
print("sigma_est: " + str(np.round(MLE_Estimate[2],3)))
```

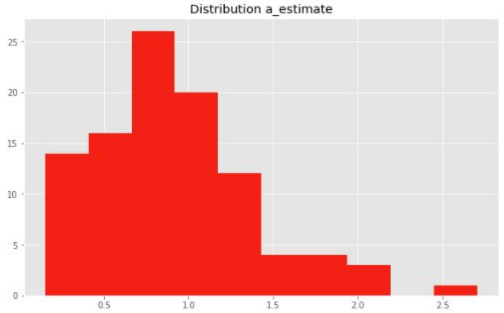
```
a_est: 0.551
b_est: 0.031
sigma_est: 0.01
```

Python Code

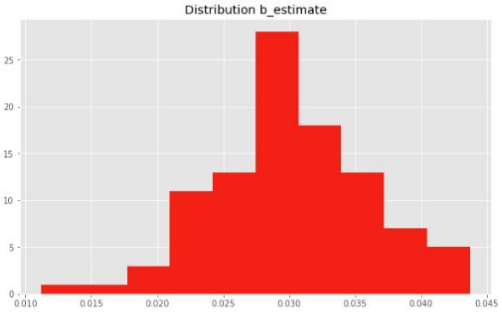
If you simulate several paths, you will remark that b and sigma estimates are quite stable contrary to a, which can then be more difficult to estimate with a strong accuracy in practice.

```python
# Model parameters
r0 = 0.02  # Initial short rate
a = 0.5    # Mean reversion speed
b = 0.03  # Long-term mean
sigma = 0.01    # Volatility
T = 10       # Time horizon
num_steps = 10000  # Number of steps
num_paths = 100   # Number of paths

a_est = []
b_est = []
sigma_est = []
for i in range(num_paths):
    simulated_rates = vasicek(r0, a, b, sigma, T, num_steps, 1)
    LS_Estimate = Vasicek_LS(simulated_rates, T / num_steps)
    a_est.append(LS_Estimate[0])
    b_est.append(LS_Estimate[1])
    sigma_est.append(LS_Estimate[2])
```

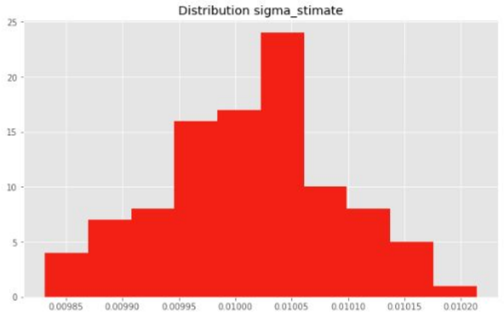
```python
plt.figure(figsize=(10, 6))
plt.title('Distribution a_estimate')
plt.hist(a_est,bins = 10);
```




```python
plt.figure(figsize=(10, 6))
plt.title('Distribution b_estimate')
plt.hist(b_est,bins = 10);
```




```python
plt.figure(figsize=(10, 6))
plt.title('Distribution sigma_stimate')
plt.hist(sigma_est, bins = 10);
```



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