American Option Pricing with Binomial Tree

American Options

The buyer of an **American Option** has the **right to exercise** the option **at any time before and including the maturity date** of the option.

Pricing of American vs European Options

$\{t_0 = 0, ..., t_i = t \cdot \frac{i}{n}, ..., t_n = t\}$ is a partition of time between 0 and t.

We assume that there is no arbitrage.

European options

The price V of an option is equal to the discounted expectation of its final payoff under the risk neutral probability Q:

$$V_{t_i} = e^{-r\cdot(t_n - t_i)} \cdot E^Q_{t_i}(V_{t_n})$$

American options

We don't have such simple expression for American options.

But we can easily calculate the price with a **backward recursion**.

Pricing of American Options With Backward Recursion

We consider an American option.

Its price is V, the underlying asset is S, and the payoff of the option is $\phi(S)$.

t_n

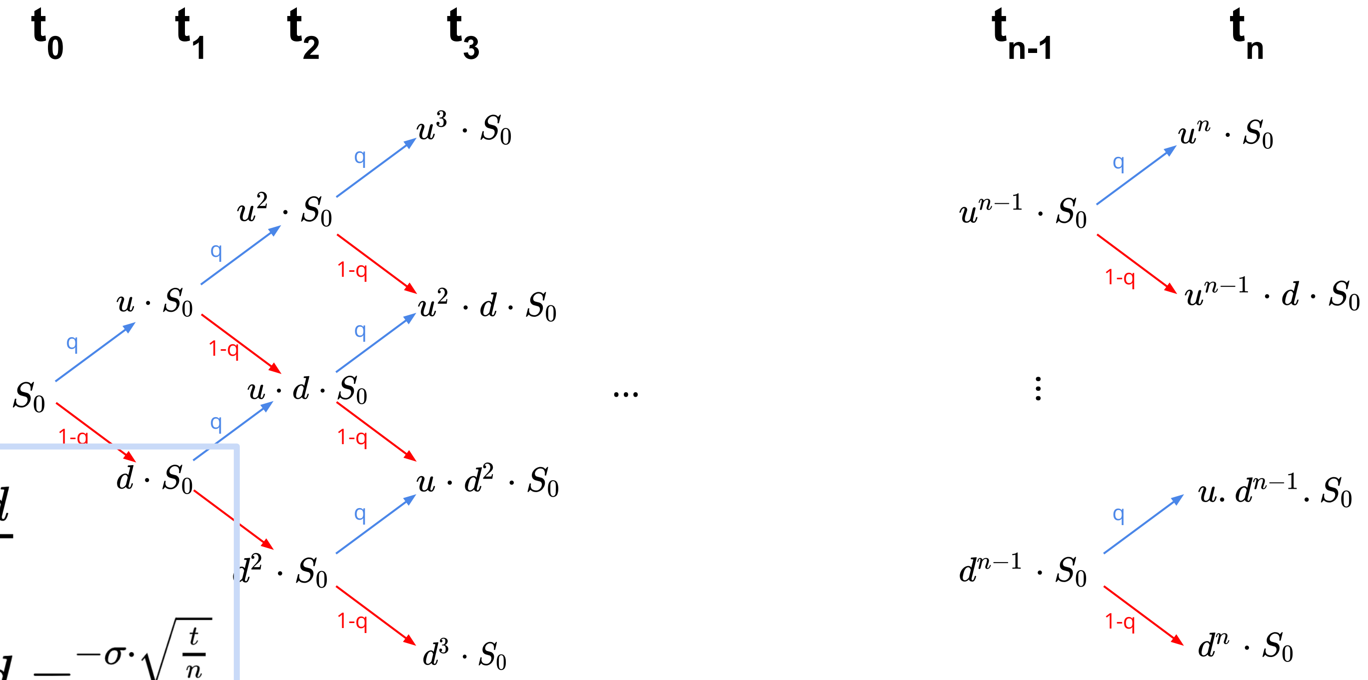
$$V_{t_n} = \phi(S_{t_n})$$

t_{n-1} The option's holder decides to exercise the option if the value of exercising the option is higher than the value of keeping it.

$$V_{t_{n-1}} = \max\left(\phi(S_{t_{n-1}}), e^{-r\cdot(t_n - t_{n-1})} \cdot E_{t_{n-1}}(\phi(S_{t_n}))\right)$$

Value of exercising the option — Value of keeping the option

Pricing of American Options in a Binomial Model



$$q = \frac{e^{r\cdot\frac{t}{n}} - d}{u - d}$$

$$u = e^{\sigma\cdot\sqrt{\frac{t}{n}}}, d = {}^{-\sigma\cdot\sqrt{\frac{t}{n}}}$$

Pricing of American Options in a Binomial Model

t_j **t_{n-1}** **t_n**

$$V_{n,n} = \phi(u^n \cdot S_0)$$

$$V_{n-1,n-1} = \max\left(\phi(u^{n-1} \cdot S_0), e^{-r\cdot\frac{t}{n}} \cdot (q \cdot V_{n,n} + (1-q) \cdot V_{n-1,n})\right)$$

$$V_{n-1,n} = \phi(u^{n-1} \cdot d \cdot S_0)$$

$$V_{i,j} = \max\left(\phi(u^i \cdot d^{j-i} \cdot S_0), e^{-r\cdot\frac{t}{n}} \cdot (q \cdot V_{i+1,j+1} + (1-q) \cdot V_{i,j+1})\right)$$

⋮

⋮

$$V_{1,n} = \phi(u \cdot d^{n-1} \cdot S_0)$$

$$V_{0,n-1} = \max\left(\phi(d^{n-1} \cdot S_0), e^{-r\cdot\frac{t}{n}} \cdot (q \cdot V_{1,n} + (1-q) \cdot V_{0,n})\right)$$

$$V_{0,n} = \phi(d^n \cdot S_0)$$

Pricing of American Options in a Binomial Model

Boundary condition:

$$\text{For } i \in [0, n]:$$
$$S_{i,n} = u^i \cdot d^{n-i} \cdot S_0$$
$$V_{i,n} = \phi(S_{i,n})$$

Backward recursion:

$$\text{For } j \in [n-1, 0]:$$
$$\text{For } i \in [0, j]:$$
$$S_{i,j} = u^i \cdot d^{j-i} \cdot S_0$$
$$V_{i,j} = \max\left(\phi(S_{i,j}), e^{-r\cdot\frac{t}{n}} \cdot (q \cdot V_{i+1,j+1} + (1-q) \cdot V_{i,j+1})\right)$$

American vs European Option Price

At maturity, the price of the American option is equal to the price of the corresponding European option.

The American option owner has more right than the European option one, as he can exercise the option at any time and we can easily show by recursion that at any date:

$$\text{American Option Price} \geq \text{European Option Price}$$

American vs European Option Price

For call options, the price of American and European options are the same when the risk-free interest rate is positive and we assume no dividend payments.

There is no benefit for the American call option holder to exercise the option before the expiry. The price of the option at t is always higher than the exercise price at t:

Jensen inequality, $t<T$:

$$E_t\left((S_T - K)^+\right) > \left(E_t(S_T) - K\right)^+$$

$$e^{-r\cdot(T-t)} \cdot E_t\left((S_T - K)^+\right) > \Big(\underbrace{e^{-r\cdot(T-t)} \cdot E_t(S_T)}_{= S_t} \underbrace{- e^{-r\cdot(T-t)} \cdot K}_{\geq -K}\Big)^+$$

$$\underbrace{e^{-r\cdot(T-t)} \cdot E_t\left((S_T - K)^+\right)}_{\text{Price of the Option at t}} > \underbrace{(S_t - K)^+}_{\text{Exercise Price at t}}$$

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